

# Pricing Deflation Risk with US Treasury Yields\*

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## Abstract

We use an arbitrage-free term structure model with spanned stochastic volatility to determine the value of the deflation protection option embedded in Treasury inflation-protected securities. The model accurately prices the deflation protection option prior to the financial crisis when its value was near zero; at the peak of the crisis in late 2008 when deflationary concerns spiked sharply; and in the post-crisis period. During 2009, the average value of this option at the 5-year maturity was 41 basis points on a par-yield basis. The option value is shown to be closely linked to overall market uncertainty as measured by the VIX, especially during and after the 2008 financial crisis.

**JEL classification:** E43, E47, G12, G13

## 1. Introduction

The US Treasury first issued inflation-indexed bonds, which are now commonly known as Treasury inflation-protected securities (TIPS), in 1997. TIPS provide inflation protection since their coupons and principal payments are indexed to the headline Consumer Price Index (CPI) produced by the Bureau of Labor Statistics.<sup>1</sup> Importantly, TIPS also provide

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1 The actual indexation has a lag structure since the Bureau of Labor Statistics publishes price index values with a 1-month lag; that is, the index for a given month is released in the middle of the subsequent month. The reference CPI is thus set to be a weighted average of the CPI for the second and third months prior to the month of maturity. See [Gürkaynak, Sack, and Wright \(2010\)](#) for a detailed discussion as well as [Campbell, Shiller, and Viceira \(2009\)](#) for an overview of inflation-indexed bonds.

some protection against price deflation since their principal payments are not permitted to decrease below their original par value.

This deflation protection option has received limited attention in the literature, most likely since it has not been of much value in the US inflationary environment since 1997. However, the sharp drops in price indexes during the financial crisis that started in the fall of 2008 increased deflationary concerns markedly, thus providing further motivation for examining the value of this protection. Two recent papers have used different arbitrage-free term structure models to assess the values of these deflation protection options.

Grishchenko, Vanden, and Zhang (2013) used a Gaussian affine model whose two factors are nominal Treasury rates and the inflation rate observed at the monthly frequency. They found that the option value is close to zero for most months, except for the deflationary periods observed in 2003–04 and in 2008–09. They calculate the maximum observed option value in December 2008 to be roughly forty-five basis points of TIPS par value. On a par-yield basis, assuming a duration of 4 years for a 5-year TIPS, this translates into a yield spread of approximately ten basis points.

Christensen, Lopez, and Rudebusch (2012) used a “yields-only” approach based on a Gaussian affine model developed by Christensen, Lopez, and Rudebusch (2010, henceforth CLR) to value these deflation protection options. That model used four factors to capture the joint dynamics of the nominal and real Treasury yield curves. The first three factors can be characterized as the level, slope, and curvature of the nominal yield curve, while the fourth factor can be characterized as the level of the real yield curve. The authors found that the option value, measured as a par-yield spread between two TIPS of similar remaining maturity but of differing vintages, reached a maximum of almost eighty basis points in December 2008 for TIPS maturing in 2013. While the model-implied option value is highly correlated with the observed TIPS yield spread chosen as a proxy for the deflation protection option value, the implied values are mainly lower than the observed values. The authors suggest that this shortcoming could be addressed by incorporating stochastic volatility (SV) into the model in the hope of better characterizing the lower tail of the model-implied distribution of inflation outcomes.

In this paper, we modify the latter model of nominal and real Treasury yields to incorporate spanned SV.<sup>2</sup> In particular, the volatility dynamics are specified to be driven by the nominal and real level factors in the model. Using the same Treasury yield data, the SV model exhibits similar in-sample fit and out-of-sample forecast performance relative to the constant volatility (CV) model. Specifically, the two models’ transformations of their conditional mean specifications into such objects of interest as 5-year inflation expectations and inflation risk premiums exhibit similar dynamics. In contrast, and more importantly for valuing the TIPS deflation protection option, the models exhibit important differences related to the transformations of their conditional volatility specification into conditional distributions of headline CPI changes.

In particular, the 1-year deflation probability forecasts generated by the SV model are generally higher than those generated by the CV model. As might be expected, the differing deflation probabilities lead to clear differences in the model-implied values of the TIPS deflation protection option. Following Wright (2009), we measure the value of this option as the yield spread between pairs of TIPS with similar maturities, but differing degrees of accumulated inflation protection; one was recently issued and has very little accrued inflation

2 Adrian and Wu (2010) also propose a model of nominal and real Treasury yields with spanned SV. In related research, Haubrich, Pennacchi, and Ritchken (2012) and Fleckenstein, Longstaff, and Lustig (2013) build models of inflation swap rates with spanned SV.

compensation (i.e., its deflation protection option is almost at-the-money), while the other is seasoned with plenty of accrued inflation compensation leaving its deflation protection option far out-of-the-money. This spread therefore is a proxy for the value of the embedded deflation protection option. As we show, the SV model generates a yield spread that more directly captures this proxy spread in the last few months of 2008 and into 2009. In fact, while both sets of model-implied spreads have correlations of nearly 0.9 with the proxy spread, the SV model has a root mean squared error over 2009 of 9.5 basis points when compared with 28.7 basis points for the CV model.

Turning to the SV model's pricing over time, in 2008 before the Lehman bankruptcy, the SV model-implied value of the TIPS deflation protection option at the 5-year maturity was 6.8 basis points. During the height of the crisis period in late 2008, that value jumped to 89.1 basis points as deflationary concerns rose markedly during the sharp economic contraction. For 2009 as a whole, the average value of this option was forty-one basis points on a par-yield basis, and that average value declined to 19.5 for 2010. These results suggest that the SV model is well equipped to measure and price deflation risk within the Treasury market, and thus it should be well placed to price the inflation derivatives increasingly being traded in the USA.<sup>3</sup>

With these estimates of the price of the embedded deflation protection option in hand, we examine the market factors that might influence its value. The empirical challenge is to assess what part of this deflation option's value reflects deflation fears associated with general economic uncertainty and what part is a reflection of market illiquidity and limits to arbitrage.<sup>4</sup> Our primary explanatory factor to account for the former is the VIX options-implied volatility index, which represents near-term uncertainty about the general stock market and is widely used as a gauge of investor risk aversion. We also include variables that gauge market illiquidity, such as the economy-wide market illiquidity measure introduced by Hu, Pan, and Wang (2013, henceforth HPW).

Our empirical results suggest that general economic uncertainty as reflected in the VIX is the main factor determining the deflation option value, accounting for about 65% of its observed variation. However, liquidity effects also play a role, particularly before and during the 2008–09 financial crisis. Based on these results, we use our modeling structure to produce liquidity-adjusted deflation probabilities using just the economic uncertainty component. The adjusted values suggest that illiquidity effects do not prevent us from drawing correct inference about when deflation risk is relevant, but they bias the assessment of the severity of the deflation risk. Further research into this important aspect of TIPS pricing and liquidity premiums is needed.

The paper is structured as follows. Section 2 introduces the general theoretical framework for inferring deflation dynamics from nominal and real Treasury yield curves and details our proposed methodology for deriving the model-implied value of the deflation protection option. Section 3 describes the CV and SV specifications of our term structure model. Section 4 contains the data description and the empirical results for the two models, while Section 5 is dedicated to an analysis of the models' inflation distributions. Section 6 focuses on the risk of deflation and its implications for the price of the deflation protection

3 See [Christensen and Gillan \(2012\)](#) for further discussion of US inflation swaps and related liquidity issues. [Kitsul and Wright \(2013\)](#) examine inflation caps and floors to derive option-implied inflation probability density functions.

4 TIPS liquidity has been a concern historically and particularly at the peak of the financial crisis; see [CLR, Pflueger and Viceira \(2013\)](#), and [Fleckenstein, Longstaff, and Lustig \(2014\)](#) for detailed discussions.

option, while Section 7 analyzes the drivers of the model-implied deflation option values. Section 8 concludes and provides directions for future research. The appendices contain additional technical details and results for alternative SV specifications.

## 2. Pricing Deflation Risk with Treasuries and TIPS

In this section, we provide the theoretical foundation for the framework we use to price deflation protection options.

### 2.1 Deriving Market-Implied Inflation Expectations and Risk Premiums

An arbitrage-free term structure model can be used to decompose the difference between nominal and real Treasury yields, also known as the breakeven inflation (BEI) rate, into the sum of inflation expectations and an inflation risk premium. We follow Merton (1974) and assume a continuum of nominal and real zero-coupon Treasury bonds exist with no frictions to their continuous trading. The economic implication of this assumption is that the markets for inflation risk are complete in the limit. Define the nominal and real stochastic discount factors, denoted  $M_t^N$  and  $M_t^R$ , respectively. The no-arbitrage condition enforces a consistency of pricing for any security over time. Specifically, the price of a nominal bond that pays one dollar in  $\tau$  years and the price of a real bond that pays one unit of the defined consumption basket in  $\tau$  years must satisfy the conditions that

$$P_t^N(\tau) = E_t^P \left[ \frac{M_{t+\tau}^N}{M_t^N} \right] \quad \text{and} \quad P_t^R(\tau) = E_t^P \left[ \frac{M_{t+\tau}^R}{M_t^R} \right], \quad (1)$$

where  $P_t^N(\tau)$  and  $P_t^R(\tau)$  are the observed prices of the zero-coupon, nominal and real bonds for maturity  $\tau$  on day  $t$  and  $E_t^P[\cdot]$  is the conditional expectations operator under the real-world (or  $P$ -) probability measure. The no-arbitrage condition also requires a consistency between the prices of real and nominal bonds such that the price of the consumption basket, denoted as the overall price level  $\Pi_t$ , is the ratio of the nominal and real stochastic discount factors:

$$\Pi_t = \frac{M_t^R}{M_t^N}. \quad (2)$$

We assume that the nominal and real stochastic discount factors have the standard dynamics given by

$$dM_t^N / M_t^N = -r_t^N dt - \Gamma_t' dW_t^P, \quad (3)$$

$$dM_t^R / M_t^R = -r_t^R dt - \Gamma_t' dW_t^P, \quad (4)$$

where  $r_t^N$  and  $r_t^R$  are the instantaneous, risk-free nominal and real rates of return, respectively, and  $\Gamma_t$  is a vector of premiums on the risks represented by  $W_t^P$ . By Ito's lemma, the dynamic evolution of  $\Pi_t$  is given by

$$d\Pi_t = (r_t^N - r_t^R)\Pi_t dt. \quad (5)$$

Thus, with the absence of arbitrage, the instantaneous growth rate of the price level is equal to the difference between the instantaneous nominal and real risk-free rates. Correspondingly, we can express the stochastic price level at time  $t + \tau$  as

$$\Pi_{t+\tau} = \Pi_t e^{\int_t^{t+\tau} (r_s^N - r_s^R) ds}. \quad (6)$$

The relationship between the yields and inflation expectations can be obtained by decomposing the price of the nominal bond as follows:

$$P_t^N(\tau) = E_t^P \left[ \frac{M_{t+\tau}^N}{M_t^N} \right] = E_t^P \left[ \frac{M_{t+\tau}^R / \Pi_{t+\tau}}{M_t^R / \Pi_t} \right] = E_t^P \left[ \frac{M_{t+\tau}^R}{M_t^R} \frac{\Pi_t}{\Pi_{t+\tau}} \right] \quad (7)$$

$$= E_t^P \left[ \frac{M_{t+\tau}^R}{M_t^R} \right] \times E_t^P \left[ \frac{\Pi_t}{\Pi_{t+\tau}} \right] + \text{cov}_t^P \left[ \frac{M_{t+\tau}^R}{M_t^R}, \frac{\Pi_t}{\Pi_{t+\tau}} \right] \quad (8)$$

$$= P_t^R(\tau) \times E_t^P \left[ \frac{\Pi_t}{\Pi_{t+\tau}} \right] \times \left( 1 + \frac{\text{cov}_t^P \left[ \frac{M_{t+\tau}^R}{M_t^R}, \frac{\Pi_t}{\Pi_{t+\tau}} \right]}{E_t^P \left[ \frac{M_{t+\tau}^R}{M_t^R} \right] \times E_t^P \left[ \frac{\Pi_t}{\Pi_{t+\tau}} \right]} \right). \quad (9)$$

Converting this price into a yield-to-maturity using

$$y_t^N(\tau) = -\frac{1}{\tau} \ln P_t^N(\tau) \quad \text{and} \quad y_t^R(\tau) = -\frac{1}{\tau} \ln P_t^R(\tau), \quad (10)$$

we obtain

$$y_t^N(\tau) = y_t^R(\tau) + \pi_t^e(\tau) + \phi_t(\tau), \quad (11)$$

where the market-implied average rate of inflation expected at time  $t$  for the period from  $t$  to  $t + \tau$  is

$$\pi_t^e(\tau) = -\frac{1}{\tau} \ln E_t^P \left[ \frac{\Pi_t}{\Pi_{t+\tau}} \right] = -\frac{1}{\tau} \ln E_t^P \left[ e^{-\int_t^{t+\tau} (r_s^N - r_s^R) ds} \right] \quad (12)$$

and the associated inflation risk premium for the same time period is

$$\phi_t(\tau) = -\frac{1}{\tau} \ln \left( 1 + \frac{\text{cov}_t^P \left[ \frac{M_{t+\tau}^R}{M_t^R}, \frac{\Pi_t}{\Pi_{t+\tau}} \right]}{E_t^P \left[ \frac{M_{t+\tau}^R}{M_t^R} \right] \times E_t^P \left[ \frac{\Pi_t}{\Pi_{t+\tau}} \right]} \right). \quad (13)$$

## 2.2 The Value of the Deflation Protection Embedded in TIPS

The primary focus of this paper is the value of the deflation protection option embedded in TIPS and how, during the financial crisis of 2008 and 2009, it affected the relative prices of pairs of TIPS differentiated only by their accrued inflation compensation. Under standard inflationary conditions with inflation rates around 2%, the value of the deflation protection option should not play an important role in TIPS pricing since the probability of having negative net accrued inflation compensation at maturity is negligible; that is, the option should be well out-of-the-money. However, at the peak of the financial crisis in the fall of 2008, neither the perceived nor the priced probability of deflation were negligible as we show in Section 6. Under these circumstances, a wedge can develop between the prices of seasoned TIPS with a significant amount of accrued inflation compensation and recently issued on-the-run TIPS, which have no cumulated inflation compensation. As suggested by Wright (2009), this wedge is a proxy for the value of the TIPS deflation protection option.

To examine the ability of the proposed models to price these deflation protection options, we use the models' implied yield curves and deflation probabilities. We calculate the deflation protection option values by comparing the prices of a newly issued TIPS without any accrued inflation compensation and a seasoned TIPS with sufficient accrued inflation

compensation under the risk-neutral (or  $Q$ -) pricing measure. First, consider a hypothetical seasoned TIPS with  $T$  years remaining to maturity that pays an annual coupon  $C$  semi-annually. Assume this bond has accrued sufficient inflation compensation so it is nearly impossible to reach the deflation floor before maturity. Under the risk-neutral pricing measure, the par-coupon bond satisfying these criteria has a coupon rate determined by the equation

$$\sum_{i=1}^{2T} \frac{C}{2} E_t^Q \left[ e^{-\int_t^{t_i} r_s^R ds} \right] + E_t^Q \left[ e^{-\int_t^T r_s^R ds} \right] = 1. \quad (14)$$

The first term is the sum of the present value of the  $2T$  coupon payments using the model's fitted real yield curve at day  $t$ . The second term is the discounted value of the principal payment. The coupon payment of the seasoned bond that solves this equation is denoted as  $C_S$ .

Next, consider a new TIPS with no accrued inflation compensation with  $T$  years to maturity. Since the coupon payments are not protected against deflation, the difference is in accounting for the deflation protection on the principal payment. For this bond, the pricing equation has an additional term; that is,

$$\sum_{i=1}^{2T} \frac{C}{2} E_t^Q \left[ e^{-\int_t^{t_i} r_s^R ds} \right] + E_t^Q \left[ \frac{\Pi_T}{\Pi_t} \cdot e^{-\int_t^T r_s^N ds} \mathbf{1}_{\left\{ \frac{\Pi_T}{\Pi_t} > 1 \right\}} \right] + E_t^Q \left[ 1 \cdot e^{-\int_t^T r_s^N ds} \mathbf{1}_{\left\{ \frac{\Pi_T}{\Pi_t} \leq 1 \right\}} \right] = 1. \quad (15)$$

The first term is the same as before. The second term represents the present value of the principal payment conditional on a positive net change in the price index over the bond's maturity; that is,  $\frac{\Pi_T}{\Pi_t} > 1$ . Under this condition, full inflation indexation applies, and the price change  $\frac{\Pi_T}{\Pi_t}$  is placed within the expectations operator. The third term represents the present value of the floored TIPS principal conditional on accumulated net deflation; that is, when the price level change is below one,  $\frac{\Pi_T}{\Pi_t}$  is replaced by a value of 1 to provide the promised deflation protection.

Since

$$\frac{\Pi_T}{\Pi_t} = e^{\int_t^T (r_s^N - r_s^R) ds}, \quad (16)$$

the equation can be rewritten as

$$\sum_{i=1}^{2T} \frac{C}{2} E_t^Q \left[ e^{-\int_t^{t_i} r_s^R ds} \right] + E_t^Q \left[ e^{-\int_t^T r_s^R ds} \right] + \left[ E_t^Q \left[ e^{-\int_t^T r_s^N ds} \mathbf{1}_{\left\{ \frac{\Pi_T}{\Pi_t} \leq 1 \right\}} \right] - E_t^Q \left[ e^{-\int_t^T r_s^R ds} \mathbf{1}_{\left\{ \frac{\Pi_T}{\Pi_t} \leq 1 \right\}} \right] \right] = 1, \quad (17)$$

where the last term on the left-hand side represents the net present value of the deflation protection of the principal in the TIPS contract. The par-coupon yield of a new hypothetical TIPS that solves this equation is denoted as  $C_0$ . The difference between  $C_S$  and  $C_0$  is a measure of the advantage of being at the inflation adjustment floor for a newly issued TIPS and thus of the value of the embedded deflation protection option.

### 3. Models of Nominal and Real Treasury Yield Curves

Given the theoretical framework introduced in the previous section, we briefly summarize the affine term structure model of nominal and real Treasury yields with CV developed

by CLR and then introduce the modified version with stochastic yield volatility. We emphasize that even though the models are not formulated using the canonical form of affine term structure models introduced by Dai and Singleton (2000), both models can be viewed as restricted versions of their respective canonical model. Furthermore, it can be noted that most of the restrictions imposed are motivated by a desire to generate a factor loading structure in the zero-coupon bond yield functions that closely matches the popular Nelson and Siegel (1987) model and hence obtain models which are well identified and easy to estimate.

### 3.1 The CV Model

The joint four-factor CV model of nominal and real yields is a direct extension of the three-factor, arbitrage-free Nelson–Siegel (AFNS) model developed by Christensen, Diebold, and Rudebusch (2011, henceforth CDR) for nominal yields. In the CV model, the state vector is denoted by  $X_t = (L_t^N, S_t, C_t, L_t^R)$ , where  $L_t^N$  is the level factor for nominal yields,  $S_t$  is the common slope factor,  $C_t$  is the common curvature factor, and  $L_t^R$  is the level factor for real yields. The instantaneous nominal and real risk-free rates are defined as:

$$r_t^N = L_t^N + S_t, \quad (18)$$

$$r_t^R = L_t^R + \alpha^R S_t. \quad (19)$$

Note that the differential scaling of the real rates to the common slope factor is captured by the parameter  $\alpha^R$ . To preserve the Nelson–Siegel factor loading structure in the yield functions, the risk-neutral (or  $\mathcal{Q}$ -) dynamics of the state variables are given by the stochastic differential equations:

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} dt + \Sigma \begin{pmatrix} dW_t^{L^N, \mathcal{Q}} \\ dW_t^{S, \mathcal{Q}} \\ dW_t^{C, \mathcal{Q}} \\ dW_t^{L^R, \mathcal{Q}} \end{pmatrix}, \quad (20)$$

where  $\Sigma$  is the constant covariance (or volatility) matrix. Based on this specification of the  $\mathcal{Q}$ -dynamics, nominal Treasury zero-coupon bond yields preserve the Nelson–Siegel factor loading structure as

$$y_t^N(\tau) = L_t^N + \left( \frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) S_t + \left( \frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) C_t - \frac{A^N(\tau)}{\tau}, \quad (21)$$

where  $A^N(\tau)/\tau$  is a maturity-dependent yield-adjustment term. Similarly, real TIPS zero-coupon bond yields have a Nelson–Siegel factor loading structure expressed as

$$y_t^R(\tau) = L_t^R + \alpha^R \left( \frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) S_t + \alpha^R \left( \frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) C_t - \frac{A^R(\tau)}{\tau}. \quad (22)$$

Note that  $A^R(\tau)/\tau$  is another maturity-dependent yield-adjustment term. These two equations, when combined in state-space form, constitute the measurement equation needed for Kalman filter estimation.

To complete the model, we define the price of risk, which links the risk-neutral and real-world yield dynamics, using the essentially affine risk premium specification introduced by Duffee (2002). The real-world dynamics of the state variables are then expressed as

$$dX_t = K^P(\theta^P - X_t)dt + \Sigma dW_t^P, \quad (23)$$

which in its most general form can be written as

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} \kappa_{11}^P & \kappa_{12}^P & \kappa_{13}^P & \kappa_{14}^P \\ \kappa_{21}^P & \kappa_{22}^P & \kappa_{23}^P & \kappa_{24}^P \\ \kappa_{31}^P & \kappa_{32}^P & \kappa_{33}^P & \kappa_{34}^P \\ \kappa_{41}^P & \kappa_{42}^P & \kappa_{43}^P & \kappa_{44}^P \end{pmatrix} \left( \begin{pmatrix} \theta_1^P \\ \theta_2^P \\ \theta_3^P \\ \theta_4^P \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} \right) dt + \Sigma \begin{pmatrix} dW_t^{L^N, P} \\ dW_t^{S, P} \\ dW_t^{C, P} \\ dW_t^{L^R, P} \end{pmatrix}. \quad (24)$$

This equation is the transition equation used in the Kalman filter estimation.

### 3.2 The SV Model

Financial time series, such as interest rates and bond yields, have been shown to have time-varying volatility, which is a feature not often incorporated into arbitrage-free term structure models; see Andersen and Benzoni (2010) for further discussion. To address this concern, Christensen, Lopez, and Rudebusch (2014a) develop a general class of AFNS models that incorporate spanned SV. To distinguish between the various types of models, we use the notation outlined in Dai and Singleton (2000) for classifying affine term structure models, such that the CV model is within the  $A_0(4)$  class of models that do not have volatility dynamics. As detailed in Christensen, Lopez, and Rudebusch (2014a), there are several possible volatility specifications within their three-factor framework, and clearly, the introduction of the fourth factor within the CLR model generates an even larger set of possible specifications.

For this paper, we chose an  $A_2(4)$  volatility specification that incorporates SV based on the nominal and real level factors.<sup>5</sup> This choice was motivated by a desire to focus on the longer maturity TIPS yields, since observable proxies for the value of the TIPS deflation protection option are most available near the 5-year maturity point. For this SV model, the state vector and instantaneous risk-free rates are the same as before. To preserve the Nelson–Siegel factor loading structure and impose our volatility specification, the  $Q$ -dynamics of the state variables are given by<sup>6</sup>

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} \kappa_{L^N}^Q & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \kappa_{L^R}^Q \end{pmatrix} \left[ \begin{pmatrix} \theta_{L^N}^Q \\ 0 \\ 0 \\ \theta_{L^R}^Q \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} \right] dt \quad (25)$$

- 5 In Appendix D, we summarize results for all seven admissible extensions of the CV model with one or two spanned SV factors (i.e.,  $A_1(4)$  and  $A_2(4)$  models). Note that the results for these alternative models are qualitatively similar, although quantitatively worse than the  $A_2(4)$  specification we focus on here.
- 6 While the modeling framework allows for the two level factors to directly affect the volatility of the common slope and curvature factors, we fix the associated volatility sensitivity parameters to zero as in Christensen, Lopez, and Rudebusch (2014a), who report that these volatility sensitivity parameters are typically insignificant for US Treasury data. This choice leads to analytical bond pricing formulas that greatly facilitate model estimation and analysis.

$$+ \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix} \begin{pmatrix} \sqrt{L_t^N} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{L_t^R} \end{pmatrix} \begin{pmatrix} dW_t^{L^N, Q} \\ dW_t^{S, Q} \\ dW_t^{C, Q} \\ dW_t^{L^R, Q} \end{pmatrix}. \quad (26)$$

The representation of the nominal zero-coupon bond yield function becomes

$$y_t^N(\tau) = g^N(\kappa_{LN}^Q) L_t^N + \left( \frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) S_t + \left( \frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) C_t - \frac{A^N(\tau; \kappa_{LN}^Q)}{\tau}, \quad (27)$$

where  $g^N(\kappa_{LN}^Q)$  is a modified loading on the nominal level factor; see Appendix A for details. Note that the slope and the curvature factor preserve their Nelson–Siegel factor loadings exactly, although the structure of the yield-adjustment term  $A^N(\tau; \kappa_{LN}^Q)/\tau$  is different than before. Correspondingly, the real zero-coupon bond yield function is now

$$y_t^R(\tau) = g^R(\kappa_{LR}^Q) L_t^R + \alpha^R \left( \frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) S_t + \alpha^R \left( \frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) C_t - \frac{A^R(\tau; \kappa_{LR}^Q)}{\tau}, \quad (28)$$

where  $g^R(\kappa_{LR}^Q)$  is a modified loading on the real level factor and  $A^R(\tau; \kappa_{LR}^Q)/\tau$  is a modified yield-adjustment term.

To link the risk-neutral and real-world dynamics of the state variables, we here use the extended affine risk premium specification introduced by [Cheridito, Filipović, and Kimmel \(2007\)](#), as per [Christensen, Lopez, and Rudebusch \(2014a\)](#). The maximally flexible affine specification of the  $P$ -dynamics is thus

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \end{pmatrix} = \begin{pmatrix} \kappa_{11}^P & 0 & 0 & \kappa_{14}^P \\ \kappa_{21}^P & \kappa_{22}^P & \kappa_{23}^P & \kappa_{24}^P \\ \kappa_{31}^P & \kappa_{32}^P & \kappa_{33}^P & \kappa_{34}^P \\ \kappa_{41}^P & 0 & 0 & \kappa_{44}^P \end{pmatrix} \left[ \begin{pmatrix} \theta_1^P \\ \theta_2^P \\ \theta_3^P \\ \theta_4^P \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \end{pmatrix} \right] dt \quad (29)$$

$$+ \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix} \begin{pmatrix} \sqrt{L_t^N} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{L_t^R} \end{pmatrix} \begin{pmatrix} dW_t^{L^N, P} \\ dW_t^{S, P} \\ dW_t^{C, P} \\ dW_t^{L^R, P} \end{pmatrix}. \quad (30)$$

To keep the model arbitrage-free, the two level factors must be prevented from hitting the lower zero-boundary. This positivity requirement is ensured by imposing the Feller conditions under both probability measures, which in this case are four; that is,

$$\kappa_{11}^P \theta_1^P + \kappa_{14}^P \theta_4^P > \frac{1}{2} \sigma_{11}^2,$$

$$10^{-7} \cdot \theta_{LN}^Q > \frac{1}{2} \sigma_{11}^2,$$

$$\kappa_{41}^P \theta_1^P + \kappa_{44}^P \theta_4^P > \frac{1}{2} \sigma_{44}^2,$$

and

$$10^{-7} \cdot \theta_{L^R}^O > \frac{1}{2} \sigma_{44}^2.$$

Furthermore, to have well-defined processes for  $L_t^N$  and  $L_t^R$ , the sign of the effect that these two factors have on each other must be positive, which requires the restrictions that

$$\kappa_{14}^P \leq 0 \quad \text{and} \quad \kappa_{41}^P \leq 0.$$

These conditions ensure that the two square-root processes will be non-negatively correlated.<sup>7</sup>

### 3.2.a. Deflation probabilities within the SV model

Christensen, Lopez, and Rudebusch (2012) use the CV model to generate deflation probabilities at various horizons appropriate for macroeconomic and monetary policy purposes. Similarly, the SV model can be used to calculate deflation probabilities, although additional steps are necessary.

The change in the price index implied by the model's "yields-only" approach for the period from  $t$  to  $t + \tau$  is given by

$$\frac{\Pi_{t+\tau}}{\Pi_t} = e^{\int_t^{t+\tau} (r_s^N - r_s^R) ds}. \quad (31)$$

To determine whether the change in the price index over a  $\tau$ -period horizon may be below a critical level  $q$ , we are interested in the probability of the states where

$$\frac{\Pi_{t+\tau}}{\Pi_t} \leq 1 + q, \quad (32)$$

or, equivalently,

$$Y_{t,\tau} = \int_t^{t+\tau} (r_s^N - r_s^R) ds \leq \ln(1 + q). \quad (33)$$

Given that  $r_t^N = L_t^N + S_t$  and  $r_t^R = L_t^R + \alpha^R S_t$ , we are interested in the distributional properties of the process

$$Y_{0,t} = \int_0^t (r_s^N - r_s^R) ds = \int_0^t (L_s^N + S_s - L_s^R - \alpha^R S_s) ds \Rightarrow dY_{0,t} = (L_t^N + (1 - \alpha^R) S_t - L_t^R) dt. \quad (34)$$

This process is then introduced into the system of equations containing the  $P$ -dynamics of the state variables  $X_t$ .

Due to the introduction of SV into the two level factors, this system of equations no longer has Gaussian state variables. As a consequence, we must use the Fourier transform

7 Our empirical results show that the Feller condition pertaining to the real yield level factor  $L_t^R$  under the  $Q$ -measure is systematically binding, while the other three Feller conditions are never binding. Thus, it is mainly the dynamics of  $L_t^R$  that are affected by the imposition of the Feller conditions, most notably  $\sigma_{44}$ . For robustness, we analyzed the specification of the SV model without Feller conditions imposed, but found it to underperform along multiple dimensions relative to the reported SV model with Feller conditions imposed. Results for this alternative specification and analysis are available upon request.

analysis described in full generality for affine models in Duffie, Pan, and Singleton (2000), as opposed to the approach detailed in Christensen, Lopez, and Rudebusch (2012) for the CV model. The intuition of this approach is to express expectations of contingent payments in a tractable, mathematical form. By simplifying these expectations to indicator variables such as  $\mathbf{1}_{(Y_{t,\tau} \leq \ln(1+q))}$ , event probabilities are readily generated; see Appendix C for details.

### 3.3 Model Estimation

While the SV model maintains linear measurement equations, its factor dynamics are non-Gaussian as noted above, which prevents us from using some of the recently proposed estimation techniques, such as Joslin, Singleton, and Zhu (2011). Instead, the estimation of both models relies on the Kalman filter as in CLR and Christensen, Lopez, and Rudebusch (2012); that is, nominal and real zero-coupon yields are affine functions of the state variables such that

$$y_t(\tau) = -\frac{1}{\tau}B(\tau)'X_t - \frac{1}{\tau}A(\tau) + \varepsilon_t(\tau), \quad (35)$$

where  $\varepsilon_t(\tau)$  are assumed to be i.i.d. Gaussian errors. The conditional mean for multi-dimensional affine diffusion processes is given by

$$E^P[X_T|X_t] = (I - \exp(-K^P(T-t)))\theta^P + \exp(-K^P(T-t))X_t, \quad (36)$$

where  $\exp(-K^P(T-t))$  is a matrix exponential. In general, the conditional covariance matrix for affine diffusion processes is given by

$$V^P[X_T|X_t] = \int_t^T \exp(-K^P(T-s))\Sigma D(E^P[X_s|X_t])D(E^P[X_s|X_t])'\Sigma' \exp(-(K^P)'(T-s))ds. \quad (37)$$

Stationarity of the system under the  $P$ -measure is ensured if the real components of all the eigenvalues of  $K^P$  are positive, and this condition is imposed in all estimations. For this reason we can start the Kalman filter at the unconditional mean and covariance matrix.<sup>8</sup> However, the introduction of SV in the SV model implies that the factors are no longer Gaussian since their variances are now dependent on the path of the state variables. For tractability, we choose to approximate the true probability distribution of the state variables using the first and second moments described above and use the Kalman filter algorithm *as if* the state variables were Gaussian.<sup>9</sup> The state equation is given by

$$X_t = (I - \exp(-K^P\Delta t))\theta^P + \exp(-K^P\Delta t)X_{t-1} + \eta_t, \quad \eta_t \sim N(0, V_{t-1}), \quad (38)$$

- 8 In the estimation, we calculate the conditional and unconditional covariance matrices using the analytical solutions provided in Fisher and Gilles (1996), which differs from the previous studies by CLR and Christensen, Lopez, and Rudebusch (2012) that relied upon numerical approximations.
- 9 A few notable examples of papers that follow this QMLE approach include Duffee (1999), Driessen (2005), and Feldhütter and Lando (2008). An unreported simulation analysis suggests that the added bias from using the Kalman filter in estimating the SV model is modest relative to the finite-sample bias that even the CV model is subject to.

where  $\Delta t$  is the time between observations and  $V_{t-1}$  is the conditional covariance matrix given in Equation (37). In the Kalman filter estimations, the error structure is given by

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim N \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} V_{t-1} & 0 \\ 0 & H \end{pmatrix} \right],$$

where  $H$  is assumed to be a diagonal matrix of the measurement error standard deviations,  $\sigma_\varepsilon(\tau_i)$ , that are specific to each yield maturity in the data set. Furthermore, the discrete nature of the transition equation can cause the square-root processes to become negative despite the fact that the parameter sets are forced to satisfy Feller conditions and other non-negativity restrictions. Whenever this happens, we follow the literature and simply truncate those processes at zero; see [Duffee \(1999\)](#) for an example.

## 4. Empirical Analysis

In this section, we detail the data used for model estimation, describe how we arrive at a preferred specification for each model, and provide a brief comparison of the models' fit.

### 4.1 Data

In this paper, the nominal Treasury bond yields used are zero-coupon yields constructed as in [Gürkaynak, Sack, and Wright \(2007\)](#).<sup>10</sup> These yields are constructed using a discount function of the [Svensson \(1995\)](#)-type to minimize the pricing error of a large pool of underlying off-the-run Treasury bonds. As demonstrated by [Gürkaynak, Sack, and Wright \(2007\)](#), the model fits the underlying pool of bond prices extremely well. By implication, the zero-coupon yields derived from this approach constitute a very good approximation of the underlying Treasury zero-coupon yield curve. From this data set, we use eight Treasury zero-coupon bond yields with maturities of 3 months, 6 months, 1 year, 2 years, 3 years, 5 years, 7 years, and 10 years. We use weekly Friday data and limit our sample to the period from January 6, 1995, to December 31, 2010, which provides us with 835 weekly observations.<sup>11</sup> Similarly, for the real Treasury yields, we use the zero-coupon bond yields constructed with the same method used by [Gürkaynak, Sack, and Wright \(2010\)](#).<sup>12</sup> The data are available from January 1999, but due to weak liquidity in the first years of TIPS trading, we follow CLR and limit our sample to the period after 2002. We have weekly real Treasury yields from January 2, 2003, to December 31, 2010, a total of 418 observations. Since our focus is on the long-term real yields, we use the six yearly maturities from 5 to 10 years.

10 The Board of Governors of the Federal Reserve updates the data on its website at <http://www.federalreserve.gov/pubs/feds/2006/index.html>.

11 We end the sample in 2010 to avoid having to address the complex problem of respecting the zero lower bound for nominal yields, which appears to have been severe since August 2011 when the FOMC first provided explicit forward guidance for future monetary policy. To support the view that this was less critical in 2009 and 2010, we point to [Swanson and Williams \(2014\)](#), who provide evidence that medium- and long-term Treasury yields responded to economic news during those 2 years in much the same way as in the prior decades.

12 This dataset is also maintained by the Board of Governors of the Federal Reserve System at <http://www.federalreserve.gov/pubs/feds/2008/index.html>.

## 4.2 Estimation Results

To select the best-fitting specifications of each model's real-world dynamics, we use a general-to-specific modeling strategy that restricts the least significant parameter in the estimation to zero and then re-estimates the model. This strategy of eliminating the least significant coefficients is carried out down to the most parsimonious specification, which has a diagonal  $K^P$  matrix. The final specification choice is based on the values of the Akaike and Bayesian information criteria as per CLR.

For the CV model, the summary statistics of the model selection process are reported in Table I. Both information criteria are minimized by specification (9), which has a  $K^P$  matrix specified as

$$K_{CV}^P = \begin{pmatrix} \kappa_{11}^P & 0 & 0 & 0 \\ \kappa_{21}^P & \kappa_{22}^P & \kappa_{23}^P & 0 \\ 0 & 0 & \kappa_{33}^P & 0 \\ \kappa_{41}^P & \kappa_{42}^P & 0 & \kappa_{44}^P \end{pmatrix}.$$

Table II contains the estimated parameters for this specification. All the off-diagonal elements are highly significant and consistent with the empirical results reported in CLR. In terms of dynamic properties, the nominal level factor is a persistent, slowly varying process not affected by any of the other factors. The common curvature factor is also unaffected by the other factors, but is less persistent and more volatile. The common slope factor is in between these two extremes as it is less persistent than the nominal level factor and less volatile than the curvature factor. Finally, the real level factor is the least persistent factor likely due to the shorter sample of real yields.

Turning to the chosen specification of the SV model, Table III contains the summary statistics of its model selection. For reasons of parsimony, we choose to focus on the specification preferred according to the BIC with a mean-reversion matrix given by

$$K_{SV}^P = \begin{pmatrix} \kappa_{11}^P & 0 & 0 & 0 \\ 0 & \kappa_{22}^P & \kappa_{23}^P & 0 \\ 0 & 0 & \kappa_{33}^P & 0 \\ 0 & 0 & 0 & \kappa_{44}^P \end{pmatrix}.$$

Compared with the preferred specification of the CV model,  $\kappa_{21}^P$  and  $\kappa_{41}^P$  are jointly only borderline significant, while  $\kappa_{42}^P$  is not admissible.

The estimated parameters for this preferred specification are reported in Table IV. The most notable difference relative to the results for the CV model is that the nominal level factor is less persistent, while the real level factor is more persistent. Furthermore, for obvious reasons,  $\sigma_{11}$  and  $\sigma_{44}$  operate at different levels now due to the introduction of SV through the first and fourth factor. However, as we show later on, these differences do not lead to major differences in the models' first moment dynamics.

Table V contains summary statistics for the fitted errors from both models. For the nominal yields, the CV model fits the very short end of the nominal yield curve relatively better than the longer maturities in the 1- to 10-year maturity range. In contrast, the SV model provides a better in-sample fit in the 1- to 10-year maturity range, but less accurate fit for short-maturity yields. For the real yields, though, the SV model provides a significant

**Table I.** Evaluation of alternative specifications of the CV model

Thirteen alternative estimated specifications of the CV model of nominal and real Treasury bond yields are evaluated. Each specification is listed with its maximum log likelihood (Max logL), number of parameters ( $k$ ), the  $P$ -value from a likelihood ratio test of the hypothesis that the specification differs from the one directly above that has one more free parameter. The information criteria (AIC and BIC) are also reported, and their minimum values are given in boldface.

Alternative specifications	Goodness-of-fit statistics				
	Max logL	$k$	$P$ -value	AIC	BIC
(1) Unrestricted $K^P$	52,561.05	40	n.a.	-105,042.1	-104,853.0
(2) $\kappa_{24}^P = 0$	52,560.99	39	0.73	-105,044.0	-104,859.6
(3) $\kappa_{24}^P = \kappa_{32}^P = 0$	52,560.89	38	0.65	-105,045.8	-104,866.1
(4) $\kappa_{24}^P = \kappa_{32}^P = \kappa_{43}^P = 0$	52,560.76	37	0.61	-105,047.5	-104,872.6
(5) $\kappa_{24}^P = \dots = \kappa_{12}^P = 0$	52,560.58	36	0.55	-105,049.2	-104,879.0
(6) $\kappa_{24}^P = \dots = \kappa_{13}^P = 0$	52,560.52	35	0.73	-105,051.0	-104,885.6
(7) $\kappa_{24}^P = \dots = \kappa_{14}^P = 0$	52,559.97	34	0.29	-105,051.9	-104,891.2
(8) $\kappa_{24}^P = \dots = \kappa_{31}^P = 0$	52,559.40	33	0.29	-105,052.8	-104,896.8
(9) $\kappa_{24}^P = \dots = \kappa_{34}^P = 0$	52,558.84	32	0.29	<b>-105,053.7</b>	<b>-104,902.4</b>
(10) $\kappa_{24}^P = \dots = \kappa_{21}^P = 0$	52,549.96	31	<0.01	-105,037.9	-104,891.4
(11) $\kappa_{24}^P = \dots = \kappa_{42}^P = 0$	52,542.19	30	<0.01	-105,024.4	-104,882.6
(12) $\kappa_{24}^P = \dots = \kappa_{41}^P = 0$	52,533.33	29	<0.01	-105,008.7	-104,871.6
(13) $\kappa_{24}^P = \dots = \kappa_{23}^P = 0$	52,516.58	28	<0.01	-104,977.2	-104,844.8

**Table II.** Parameter estimates for the preferred CV model

The estimated parameters of the KP matrix,  $\theta^P$  vector, and diagonal  $\Sigma$  matrix are shown for the specification of the CV model preferred according to both AIC and BIC information criteria. The estimated value of  $\lambda$  is 0.5016 (0.0034), while  $\alpha^R$  is estimated to be 0.5600 (0.0056). The numbers in parentheses are the estimated parameter standard deviations. The maximum log likelihood value is 52,558.84.

$K^P$	$K_{\cdot,1}^P$	$K_{\cdot,2}^P$	$K_{\cdot,3}^P$	$K_{\cdot,4}^P$	$\theta^P$	$\Sigma$
$K_{1,\cdot}^P$	0.3483 (0.2528)	0	0	0	0.0637 (0.0045)	$\sigma_{11}$ (0.0002)
$K_{2,\cdot}^P$	1.4559 (0.4738)	0.8185 (0.1678)	-0.8148 (0.1152)	0	-0.0289 (0.0174)	$\sigma_{22}$ (0.0002)
$K_{3,\cdot}^P$	0	0	0.5416 (0.2897)	0	-0.0175 (0.0135)	$\sigma_{33}$ (0.0006)
$K_{4,\cdot}^P$	-4.1070 (0.5491)	-0.6406 (0.1874)	0	3.1116 (0.3428)	0.0372 (0.0047)	$\sigma_{44}$ (0.0001)

overall improvement in model fit relative to the CV model, which is the main cause for the large difference in likelihood values.

In the remainder of the paper, we analyze the performance of the two models in greater detail using real-time analysis that adds 1 week of additional data to the estimation sample

**Table III.** Evaluation of alternative specifications of the SV model

Nine alternative estimated specifications of the SV model of nominal and real Treasury bond yields are evaluated. Each specification is listed with its log likelihood (Max logL), number of parameters ( $k$ ), the  $P$ -value from a likelihood ratio test of the hypothesis that the specification differs from the one directly above that has one more free parameter. The information criteria (AIC and BIC) are also reported, and their minimum values are given in boldface.

Alternative specifications	Goodness-of-fit statistics				
	Max logL	$k$	$P$ -value	AIC	BIC
(1) Unrestricted $K^P$	54,479.99	38	n.a.	-108,884.0	-108,704.3
(2) $\kappa_{34}^P = 0$	54,479.99	37	1.00	-108,886.0	-108,711.1
(3) $\kappa_{34}^P = \kappa_{24}^P = 0$	54,479.85	36	0.60	-108,887.7	-108,717.5
(4) $\kappa_{34}^P = \kappa_{24}^P = \kappa_{31}^P = 0$	54,479.26	35	0.28	-108,888.5	-108,723.1
(5) $\kappa_{34}^P = \dots = \kappa_{32}^P = 0$	54,479.12	34	0.60	<b>-108,890.2</b>	-108,729.5
(6) $\kappa_{34}^P = \dots = \kappa_{21}^P = 0$	54,477.19	33	0.05	-108,888.4	-108,732.4
(7) $\kappa_{34}^P = \dots = \kappa_{41}^P = 0$	54,473.33	32	< 0.01	-108,882.7	-108,731.4
(8) $\kappa_{34}^P = \dots = \kappa_{14}^P = 0$	54,470.80	31	0.02	-108,879.6	<b>-108,733.0</b>
(9) $\kappa_{31}^P = \dots = \kappa_{23}^P = 0$	54,437.41	30	< 0.01	-108,814.8	-108,673.0

**Table IV.** Parameter estimates for the preferred SV model

The estimated parameters of the KP matrix, the  $\theta^P$  vector, and the  $\Sigma$  matrix for the preferred specification of the SV model according to the BIC. The  $Q$ -related parameters are estimated at:  $\lambda = 0.6067$  (0.0025),  $\alpha^R = 0.4397$  (0.0068),  $\theta_{LN}^Q = 32,419$  (31.67), and  $\theta_{LR}^Q = 17,846$  (47.15). The numbers in parentheses are the estimated standard deviations of the parameter estimates. The maximum log likelihood value is 54,470.80.

$K^P$	$K_{\cdot,1}^P$	$K_{\cdot,2}^P$	$K_{\cdot,3}^P$	$K_{\cdot,4}^P$	$\theta^P$	$\Sigma$
$K_{1,\cdot}^P$	1.0431 (0.4193)	0	0	0	0.0425 (0.0045)	$\sigma_{11}$ (0.0063)
$K_{2,\cdot}^P$	0	0.6711 (0.1867)	-0.6248 (0.1549)	0	-0.0118 (0.0143)	$\sigma_{22}$ (0.0003)
$K_{3,\cdot}^P$	0	0	0.6915 (0.1966)	0	-0.0076 (0.0119)	$\sigma_{33}$ (0.0007)
$K_{4,\cdot}^P$	0	0	0	1.4203 (0.1914)	0.0168 (0.0017)	$\sigma_{44}$ (0.0007)

for each model estimation; that is, each model is estimated using the sample covering the 12-year period from January 6, 1995, to January 5, 2007, and relevant model output is calculated; then, 1 week of data is added to the sample and the models are re-estimated, and another set of model output is constructed. This process is continued until the sample ends on December 31, 2010.

## 5. Analysis of the Model-Implied Inflation Distributions

In this section, we analyze the properties of the model-implied inflation distributions from the CV and SV models. First, we evaluate the models' conditional inflation

**Table V.** Summary statistics of the fitted errors

The mean and RMSE for the preferred specification of the CV and SV models are shown. All numbers are measured in basis points. The nominal yields cover the period from January 6, 1995, to December 31, 2010, while the real TIPS yields cover the period from January 3, 2003, to December 31, 2010.

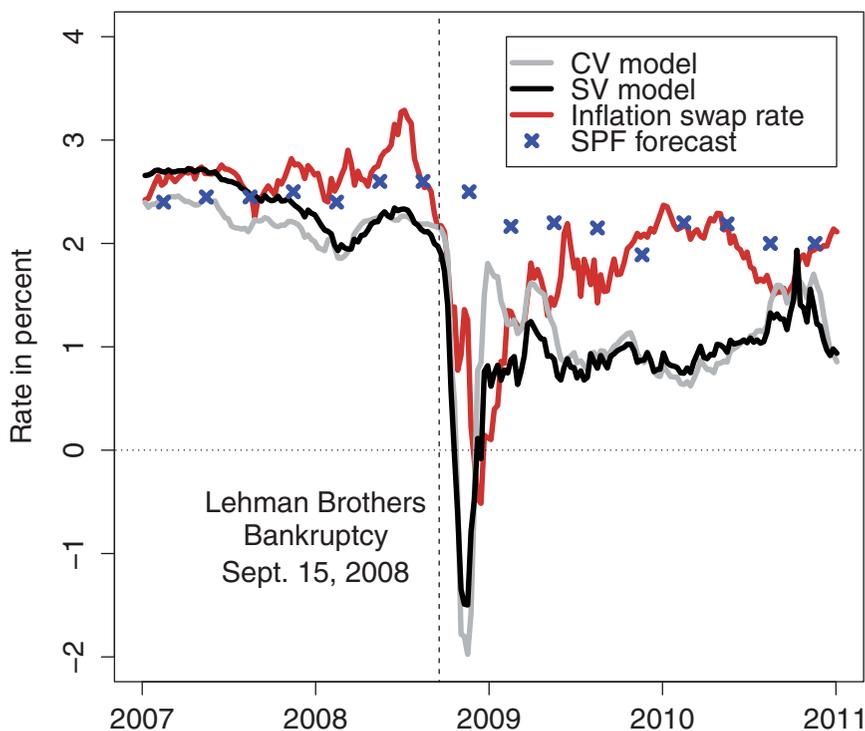
Maturity in months	CV model			SV model		
	Mean	RMSE	$\hat{\sigma}_\varepsilon(\tau_i)$	Mean	RMSE	$\hat{\sigma}_\varepsilon(\tau_i)$
Nominal yields						
3	-0.54	9.53	9.51	0.75	19.23	19.22
6	0.00	0.00	0.00	-0.17	8.23	8.24
12	1.79	5.80	5.79	0.00	0.00	0.00
24	2.22	3.98	4.00	0.46	1.56	1.56
36	0.00	0.13	0.55	0.00	0.00	0.00
60	-2.67	3.73	3.83	-0.28	1.27	1.38
84	0.08	3.37	3.66	0.24	0.59	1.14
120	9.53	12.03	12.15	-1.15	4.41	4.59
TIPS yields	Mean	RMSE	$\hat{\sigma}_\varepsilon(\tau_i)$	Mean	RMSE	$\hat{\sigma}_\varepsilon(\tau_i)$
60	-3.98	20.27	20.20	-2.04	13.59	13.59
72	-2.60	12.23	12.18	-0.51	5.87	5.86
84	-1.31	5.64	5.61	0.00	0.00	0.00
96	0.00	0.00	0.00	-0.38	4.72	4.72
108	1.35	4.94	4.90	-1.52	8.74	8.74
120	2.74	9.32	9.25	-3.32	12.35	12.35
Max logL		52,558.84			54,470.80	

expectations by comparing them with inflation swap rates and survey forecasts. Second, we highlight the models' differential estimates of the risk of tail outcomes in the form of low and high inflation. Overall, the purpose is to demonstrate the distributional limitations of models with CV, while showcasing the advantages that models with SV may provide.

### 5.1 Conditional Inflation Expectations

A key purpose of our joint models of nominal and real yields is to decompose BEI rates into inflation expectations and inflation risk premiums for further analysis. To conduct this analysis, we generate real-time, out-of-sample forecasts based on the rolling model estimation procedure described previously.

Figure 1 illustrates the models' expected inflation at the 5-year horizon as well as the 5-year zero-coupon inflation swap rate and the median of the 5-year CPI inflation forecast from the Survey of Professional Forecasters (SPFs). Similar to the inflation swap rate, the CV and SV models produce sharp declines in expected inflation shortly after the Lehman Brothers bankruptcy in September 2008, which is consistent with realized inflation; that is, headline CPI did register negative year-over-year changes during 2009 for the first time since 1955. Since the beginning of 2009, the two models suggest that medium-term inflation expectations have stabilized, but at a lower level than what prevailed prior to the financial crisis. This downward shift is consistent with the downward trend in the SPF survey



**Figure 1.** Estimated 5-year inflation expectations.

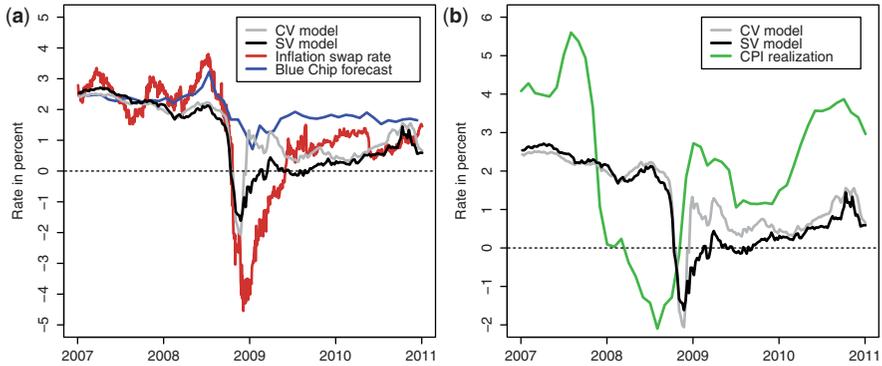
Illustration of the estimated inflation expectations at the 5-year horizon under the objective  $P$  probability measure according to the CV and SV models. Included are the 5-year zero-coupon inflation swap rate downloaded from Bloomberg and the median 5-year forecast of headline CPI inflation from the Survey of Professional Forecasters.

measure, but notably larger. Furthermore, it appears consistent with the trend in CPI realizations, which has shifted down.<sup>13</sup>

In [Figure 2\(a\)](#), the 1-year inflation forecasts from the two models are compared with the corresponding survey forecasts provided by Blue Chip and the 1-year inflation swap rate. The survey forecasts are very stable and typically favored by macroeconomists. The inflation swap rates represent financial market forecasts of inflation without correction for risk premiums, which explains their relative volatility. We note that our market-based models strike a balance in between these two alternative inflation forecasts with a tendency to be closer aligned with the inflation swap rate forecast. [Figure 2\(b\)](#) compares our model forecasts to the subsequent headline CPI realizations. As is common in the literature, both the model- and survey-based forecasts are challenged in capturing the large variation in headline CPI.<sup>14</sup> However, to compare the various forecasts in relative terms, [Table VI](#) reports

13 From the beginning of 2006 until the end of June 2008, the average annual rate of headline CPI inflation was  $\log(218.815/195.3)/2.5 = 4.5\%$ , while the average annual rate from mid-2008 through 2010 was a modest  $\log(219.179/218.815)/2.5 = 0.1\%$ .

14 See [Stock and Watson \(2007\)](#) and [Trehan \(2015\)](#) for further discussion.



**Figure 2.** One-year CPI inflation forecasts.

Panel (a) shows the 1-year inflation forecasts under the objective  $P$  probability measure from the CV and SV models with a solid gray and black line, respectively. Included are the monthly Blue Chip 1-year headline CPI inflation forecast (solid blue line) and the 1-year zero-coupon inflation swap rate downloaded from Bloomberg (solid red line). Panel (b) shows the year-over-year headline CPI inflation realizations with a solid green line and compares it the 1-year inflation forecasts from the CV and SV models shown with a solid gray and black line, respectively.

**Table VI.** Comparison of real-time CPI inflation forecasts

Summary statistics for 1-year forecast errors of headline CPI inflation in real time. The Blue Chip forecasts are mapped to the 10th of each month from January 2007 to December 2010, a total of 48 monthly forecasts. The comparable model forecast is generated on the nearest available business day prior to the Blue Chip release. A similar principle is used for the collection of the corresponding inflation swap rate forecast. The subsequent CPI realizations are year-over-year changes starting at the end of the survey month. As a consequence, the random walk forecasts equal the past year-over-year change in the CPI series as of the end of the survey month.

Model	Mean	RMSE
Random Walk	7.36	316.48
Blue Chip	5.71	209.42
Inflation swap	84.30	283.67
CV model	74.37	219.86
SV model	96.55	232.43

the results of aligning the model-generated inflation forecasts with the release dates of the Blue Chip survey and calculating the forecast errors for the 48 months from January 2007 to December 2010. In terms of matching headline CPI inflation, our two models are close behind the Blue Chip survey forecasts and are clearly better than the random walk and the 1-year inflation swap rate as measured by root mean squared fitted errors (RMSEs). This result suggests that both models are able to capture relatively well and in real time the first moment dynamics of the inflation process.

## 5.2 Tails of the Inflation Distribution

In this section, we demonstrate that there are notable differences in the models' assessment of the uncertainty about the inflation outlook. To do so, we rely on the real-time model estimates and analyze the tails of the models' conditional inflation distributions in greater detail.

In [Figure 3\(a\)](#), we focus on the risk of low inflation defined as annual inflation averaging less than 1% over the next 3 years. We note that the estimated probabilities from the CV model exhibit a very erratic pattern; before late 2008 they suggested that the risk of low inflation was entirely negligible, but suddenly the estimated probability of this risk spiked to equal one at the peak of the financial crisis, only to fall back to very low levels a few months later. This behavior is the consequence of the model's very narrow distribution for the most likely inflation outcomes.

In [Figure 3\(b\)](#), we consider the risk of high inflation defined as annual inflation averaging above 3% over the next 3 years, which historically has not been an unlikely event. Still, according to the CV model, this was an almost impossible outcome, even in 2007 and early 2008 when energy and commodity prices were near all-time highs. Importantly, during that period, it also put the risk of low inflation at near zero. This underscores the narrowness of its inflation distribution at forecast horizons relevant for monetary policy analysis.

In contrast, the variation in the estimated probabilities of these two inflation outcomes based on the SV model is more gradual. Particularly noteworthy is the fact that, once an inflation outcome has a high likelihood, its estimated probability varies only slowly consistent with the high persistence of the underlying inflation process. Furthermore, at times, both tail outcomes can have non-negligible probabilities simultaneously according to this model.

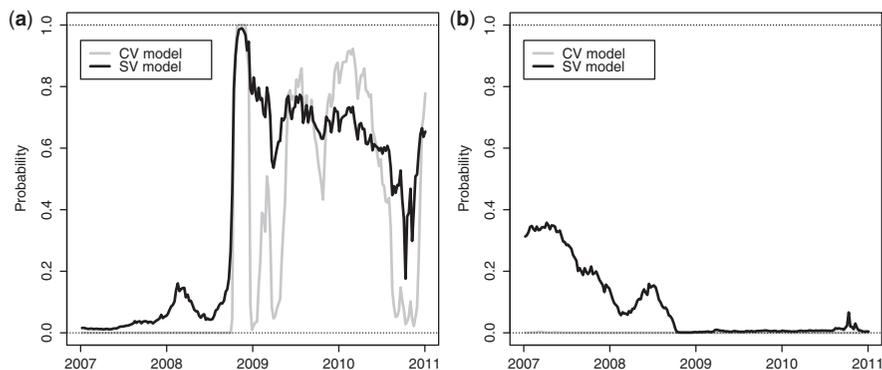
To go beyond the descriptive analysis above and perform a more rigorous test of the two models' ability to capture tail inflation risks, we focus on the risk of deflation and its implications for the value of the option protecting against deflation embedded in the TIPS contract. The remainder of the paper is devoted to this task.

## 6. Deflation Risk

In this section, we first compare the models' deflation probability forecasts before we proceed to assess their ability to value the deflation protection option embedded in TIPS.

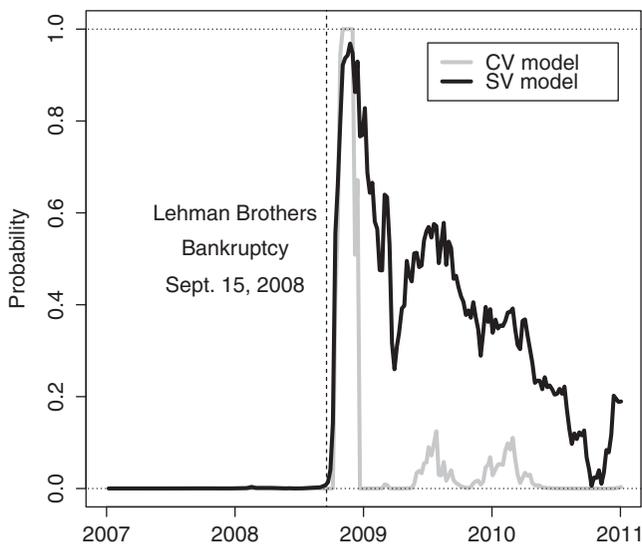
### 6.1 Deflation Probability Forecasts

To begin, [Figure 4](#) shows the models' implied objective probability forecasts of net deflation 1 year ahead. The risk of deflation in 2007 and leading up to the failure of Lehman Brothers in September 2008 was basically zero under both models. In late 2008, the models assigned a high probability to net deflation over the following 12-month period, which is consistent with the observed negative year-over-year change in headline CPI observed during these months. The SV model probabilities are markedly higher than the CV model probabilities starting at the end of 2008 through year-end 2010. These higher and more persistent probabilities are partly a reflection of slightly lower short-term expected inflation within the SV model during this period (see [Figure 2](#)), but mainly they are due to the SV model's higher conditional volatility estimates that make tail outcomes more likely as also emphasized in the previous section. Furthermore, in light of the fact that the economy did experience negative headline CPI inflation during 2009, we consider the deflation



**Figure 3.** Model-based probabilities of inflation outcomes.

Panel (a) shows the estimated probability of headline CPI inflation averaging below 1% over the next 3 years, a low inflation outcome, according to the CV and SV models. Panel (b) shows the estimated probability of headline CPI inflation averaging above 3% over the next 3 years, a high inflation outcome, according to the CV and SV models. The data series represent real-time weekly estimates under the objective  $P$  probability measure covering the period from January 6, 2007, to December 31, 2010.



**Figure 4.** Estimated 1-year deflation probabilities.

Illustration of the estimated probability of negative net inflation over the following 1-year period under the objective  $P$  probability measure according to the CV and SV models.

probability forecasts from the CV model to be low, while the forecasts from the SV model appear more reasonable with estimates in the 30–50% range through most of 2009.

## 6.2 Deflation Protection Option Values

In this section, we use our rolling estimation results to analyze the models' ability to price the deflation protection option embedded in TIPS using the methodology described in

Section 2.2. To highlight the difference between the CV and SV models in this regard, Figure 5 shows the two model-implied values of the embedded TIPS deflation protection option measured as the difference in value between a newly issued TIPS and an otherwise identical seasoned TIPS converted into par-coupon yield spreads. The shown series are synthetic, constant 5-year par-yield spreads implied by both models. The figure also shows the actually observed yield differences between seasoned and recently issued TIPS with maturities in 2013, 2014, and 2015. At each point in time, we only show the yield spread for the TIPS pair containing the most recently issued 5-year TIPS, which we refer to as the on-the-run pair, and that represents the closest observable proxy to the model-implied constant-maturity yield spread.<sup>15</sup>

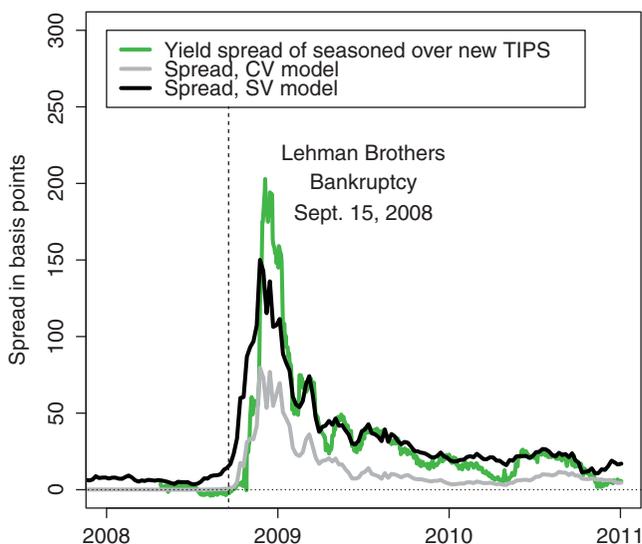
As observed by Christensen, Lopez, and Rudebusch (2012), the CV model consistently undervalues the deflation protection option even though it follows its time-variation well. The SV model is much more successful at matching the observed value of the deflation option prior to the crisis, at the peak of the crisis, as well as in the post-crisis period. Table VII shows that the SV model provides better estimates of the embedded TIPS deflation option over the sample period of April 2008 through December 2010, both in terms of mean fitted error (i.e.,  $-1.50$  basis points for the SV model relative to  $+10.49$  basis points for the CV model) and root-mean squared error (i.e.,  $19.28$  versus  $25.09$  basis points). Looking more carefully at subperiods, both models performed similarly prior to the Lehman bankruptcy in September 2008, but for the remainder of 2008, the SV model's RMSE was lower at  $50.6$  basis points when compared with the CV model's value of  $62.4$  basis points. The SV model again outperformed the CV model over the course of 2009 with an RMSE of  $33.4$  basis points relative to  $40.5$  basis points, and in 2010, the corresponding RMSE values were similar at  $8.4$  versus  $7.1$  basis points. The ability of the SV model to handle the greater volatility observed during the financial crisis, while performing as well as the CV model before and after the crisis period, is strong evidence in favor of using this model for term structure modeling, especially for interest-rate derivatives pricing and capturing the data's second-moment dynamics.

To further illustrate the relative performance of the models, we examine the fitted values of the model-implied equivalents of the yield-to-maturity for each of the TIPS in the on-the-run pair separately.<sup>16</sup> For this exercise, we match the timing of the outstanding coupons and principal for each bond exactly, although we neglect the lag in the inflation indexation since such adjustments are typically small for medium-term bonds. Specifically, we generate the net present value of the remaining bond payments using Equation (14) and the fitted real yield curve to convert the bond price into yield-to-maturity. In addition, we add the model-implied value of the deflation protection option before converting the bond price into yield-to-maturity. We explicitly control for the accrued inflation compensation in the option valuation; that is, the option will only be in the money provided that

$$\frac{\Pi_T}{\Pi_t} \leq \frac{1}{\Pi_t/\Pi_0}, \quad (39)$$

15 Specifically, from April 23, 2008, to April 22, 2009, we use the 5-year TIPS with maturity in April 2013 and the 10-year TIPS with maturity in July 2013. From April 23, 2009, to April 23, 2010, we use the 5-year TIPS with maturity in April 2014 and the 10-year TIPS with maturity in July 2014. Since April 26, 2010, we use the 5-year TIPS with maturity in April 2015 and the 10-year TIPS with maturity in July 2015.

16 We are grateful to Kenneth Singleton for suggesting this exercise.



**Figure 5.** Proxy value of the five-year TIPS deflation protection option.

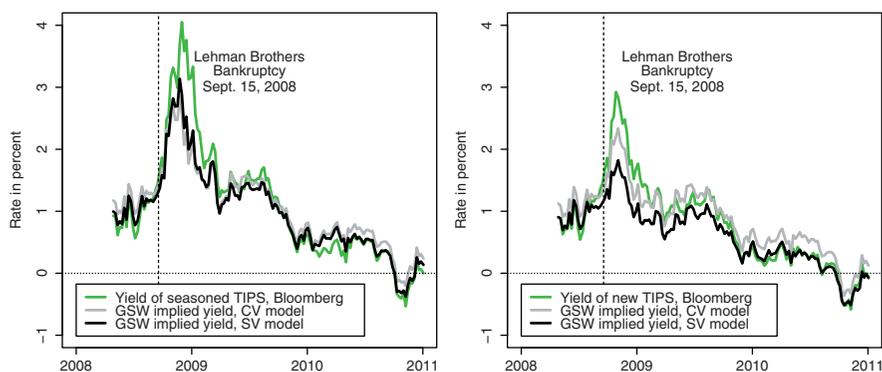
Illustration of the estimated 5-year par-coupon yield spread between a seasoned and a newly issued TIPS according to the CV and SV models. Included is also the spread in yield-to-maturity between comparable pairs of seasoned and newly issued TIPS with approximately 5 years remaining to maturity as reported by Bloomberg. As per Wright (2009), this spread can be viewed as a proxy for the value of the embedded TIPS deflation protection option. See footnote 15 for complete details on the specific nominal and real bond pairs used to generate the series.

**Table VII.** Summary statistics of pricing errors for Bloomberg data

The table reports the mean and the root mean squared pricing error of the yield-to-maturity for the seasoned and newly issued TIPS in the pair of TIPS that contains the on-the-run 5-year TIPS as reported by Bloomberg. For comparison the last row reports the comparable in-sample mean and RMSE of the 5-year TIPS yield in the Gürkaynak, Sack, and Wright (2010) data based on the full sample estimation of each model. All numbers are measured in basis points. The data are weekly covering the period from April 25, 2008, to December 31, 2010, a total of 141 observations.

TIPS yield	CV model		SV model	
	Mean	RMSE	Mean	RMSE
(a) Seasoned	1.88	38.68	15.76	39.41
(b) Newly issued	-8.61	23.75	17.26	32.32
Yield spread (a-b)	10.49	25.09	-1.50	19.28
Five-year GSW yield	9.29	28.48	4.67	18.61

where  $\Pi_t/\Pi_0$  is the index ratio as of time  $t$ ; see Appendix B for further details. Thus, for the option to be in the money, the deflation experienced over the remaining life of the bond,  $\Pi_T/\Pi_t$ , has to negate the accumulated inflation experienced since the bond's issuance.



**Figure 6.** Comparison of model-implied TIPS yields to Bloomberg data.

Comparison of the model-implied yields based on GSW data to the yield-to-maturity of the seasoned and newly issued TIPS as reported by Bloomberg.

As shown in [Figure 6](#), both models fit the bond-specific yield data relatively well outside of the peak of the financial crisis in the fall of 2008 and the early part of 2009 when TIPS liquidity was a concern. [Table VII](#) shows that the SV model does not perform as well in pricing the individual bonds as it does in capturing the spread and thus the embedded option values. For the sample period, the SV model has larger mean errors for both the seasoned and newly issued TIPS. In terms of RMSE, its value for the seasoned bonds is on par with that of the CV model, but it is much higher for the newly issued bonds. As observed in [Table V](#) regarding the models' comparative in-sample fit for the nominal and real yield curves, the relative advantage of the SV model is not obvious when examining the data's first moment dynamics, whether for the real yield curve or for individual bond yields. However, the model's ability to price the option values implicit in the spread between the on-the-run bond pairs reflects its advantage in better capturing the data's second moment dynamics. Thus, the introduction of SV into term structure models is an important extension for modeling interest rate risk and derivatives pricing.

## 7. Analysis of the Deflation Option Values

In this section, we use regression analysis to identify the determinants of the model-implied deflation option values defined as the par-bond yield spread between a seasoned and a comparable newly issued TIPS as described in the previous section.<sup>17</sup> We acknowledge that TIPS market functioning, along with the functioning of so many other financial markets, was impaired at the peak of the financial crisis, and our models do not readily account for such liquidity effects. As a consequence, we attempt to assess how much of the variation in the deflation option value during our sample period reflects outright deflation fears caused by economic uncertainty and how much could be associated with market illiquidity and

17 This analysis and the choice of variables are heavily inspired by [Christensen and Gillan \(2015\)](#), who assess the impact on frictions in the TIPS and inflation swap markets from the TIPS purchases included in the Federal Reserve's second program of large-scale asset purchases, frequently referred to as QE2, that operated from November 2010 to June 2011.

limits to arbitrage. Finally, we use the regression results to analyze the extent to which the estimated illiquidity effects may affect the assessment of the deflation risk.

### 7.1 Econometric Challenge

The correlation between states of the world with near-zero interest rates and states of the world with deflation is intuitively high, as per the Fisher equation that states that the nominal interest rate equals the real interest rate plus the rate of inflation. Unfortunately, the near-zero interest rates in the USA came about as a policy response to the freezing of financial markets. Correspondingly, in the data, poor market functioning coincides with low interest rates. Worse still, in the post-crisis period when financial conditions started to normalize, the reverse pattern was observed; that is, improvement in market functioning goes hand in hand with reduced risk of deflation. Thus, in our regressions, the deflation option value, which is our dependent variable, will tend to decline at the same time as measures of market functioning improve without the two having a causal relationship. In short, our results could likely be interpreted as indicating that the yield wedge between seasoned and newly issued TIPS was caused by limits to arbitrage, rather than reflecting true expectations for deflation.

### 7.2 Dependent Variables

For our regression analysis, our dependent variable is the synthetic par-coupon bond yield spread between a seasoned TIPS whose deflation protection option value can be assumed to be zero and a newly issued TIPS where the option is at-the-money. To be consistent with the previous analysis, we limit our focus to the 5-year deflation option value series. These estimated series from the two models are shown in Figure 7 and represent the dependent variables in our regressions.<sup>18</sup>

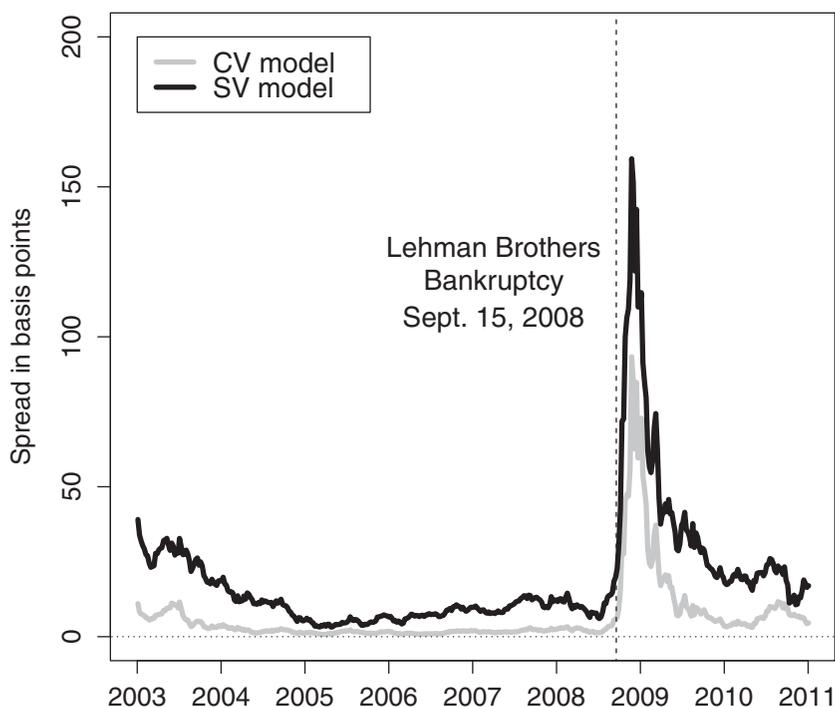
### 7.3 Explanatory Variables

In this section, we provide a brief description of the explanatory variables included in our analysis. While the other factors to be considered are supposed to capture limits to arbitrage or pricing frictions, our first and leading candidate is a measure of priced economic uncertainty, namely the VIX options-implied volatility index. It represents near-term uncertainty about the general stock market as reflected in 1-month options on the Standard and Poor's 500 stock price index and is widely used as a gauge of investor fear and risk aversion. When the price of uncertainty goes up as reflected in higher values of the VIX, the value of the TIPS deflation protection option should go up as well.

The second variable is a market illiquidity measure introduced in a recent paper by HPW.<sup>19</sup> They demonstrate that deviations in bond prices in the Treasury securities market from a fitted yield curve represent a measure of noise and illiquidity caused by limited availability of arbitrage capital. Their analysis suggests that the measure is a priced risk factor across several financial markets, which they interpret to imply that their error series

18 We emphasize that the option values in Figure 7 are based on the model parameters estimated as of the last day of the sample, unlike in prior sections in which we present results based on rolling model re-estimations. This provides a longer sample for the regression analysis that is less dominated by the financial crisis.

19 The data are publicly available at Jun Pan's website: <https://sites.google.com/site/junpan2/publications>.



**Figure 7.** Five-year deflation option values.

Illustration of the estimated 5-year deflation option values from the CV and SV models based on the parameter values at the end of the sample period. The data cover the period from January 3, 2003, to December 31, 2010.

represents an economy-wide illiquidity measure that should affect all financial markets including the market for TIPS.

The third variable considered is the yield difference between seasoned, or so-called off-the-run, Treasury securities and the most recently issued, or so-called on-the-run, Treasury security.<sup>20</sup> For each maturity segment in the Treasury bond market, the on-the-run security is typically the most traded security and therefore least penalized in terms of liquidity premiums. For our analysis, the important thing to note is that, provided there is a wide yield spread between liquid on-the-run and comparable seasoned Treasuries, we would expect a similar widening of the yield spread between comparable seasoned and newly issued TIPS.

Our fourth explanatory variable is the excess yield of AAA-rated US industrial corporate bonds over comparable Treasury yields, as per Christensen, Lopez, and Rudebusch (2014b). We use the 2-year credit spread, which strikes a balance between matching the maturity of the deflation options and focusing on a maturity at which the credit risk of AAA-rated corporate bonds is negligible. This yield spread largely reflect the premium bond investors required for being exposed to the lower trading volume and larger bid-ask

20 We focus on the most widely used 10-year maturity and construct the spread by taking the difference between the off-the-run Treasury par-coupon bond yield from the GSW (2007) database and the on-the-run Treasury par-coupon bond yield from the H.15 series at the Board of Governors.

spreads in the corporate bond market vis-à-vis the liquid Treasury bond market. Again, if such illiquidity premiums of high-quality corporate bonds are large, we could expect wider yield spreads between comparable seasoned and newly issued TIPS.

The fifth and final variable included is the weekly average of the daily trading volume in the secondary market for TIPS as reported by the Federal Reserve Bank of New York.<sup>21</sup> We use the 8-week moving average to smooth out short-term volatility. This measure should have a negative effect on the value of the deflation option provided it reflects limits to arbitrage as increases in TIPS trading volume should, in most cases, reduce mispricing.

#### 7.4 Regression Results

Table VIII reports the results of regressing the 5-year deflation option values from the CV (top panel) and SV (bottom panel) models on the explanatory variables described in the previous section.<sup>22</sup>

First, the regressions with the deflation option values derived from the SV model generally produce higher adjusted  $R^2$ s. Second, these  $R^2$  easily exceeds 80%, suggesting that our five explanatory variables are successful in capturing much of the variation in the deflation option values. Third, and most importantly, the VIX is always a highly significant explanatory variable with an estimated coefficient of the right sign and of economically meaningful size. Thus, one robust finding is that financial market uncertainty as measured by the VIX is a key component of TIPS deflation option value. Furthermore, and not surprisingly, the measure of financial market illiquidity introduced by HPW also consistently has a high explanatory power with a positive sign for its estimated coefficient. This result suggests that at least part of the variation in our deflation option values reflects financial market illiquidity. In addition, the other three measures of market liquidity and market functioning individually have the expected sign, but when combined with the VIX and the HPW measure, their added explanatory power is low. Since we have not corrected the TIPS yields used in model estimation for any liquidity effects, these results were to be expected. Still, we consider the high significance of the VIX in these regressions a strong indication that the model-implied deflation protection option values are real and not a spurious artifact caused by changes in TIPS market liquidity.

#### 7.5 Liquidity-Adjusted Deflation Probabilities

In this section, we use the above regression results to produce model-implied deflation probabilities that are adjusted for the estimated TIPS liquidity effects.<sup>23</sup> We proceed by separating the 5-year deflation option value from the SV model into an economic uncertainty component and a liquidity component, based on regression (7) in the bottom panel of Table VIII. The VIX multiplied by its estimated regression coefficient represents an estimate of the uncertainty component, here referred to as the liquidity-adjusted deflation option values and shown in Figure 8 with a comparison to the estimated 5-year deflation option values. We note that TIPS illiquidity effects played a material role in the early part of the

21 The data are available at: <http://www.newyorkfed.org/markets/statrel.html>.

22 The results reported in Table VIII are robust to using other maturities and sample periods. We also conducted this analysis using the observed difference between seasoned and recently issued TIPS securities and achieved qualitatively similar results.

23 We thank an anonymous referee for suggesting this exercise. Please note that we limit our focus to the SV model, but a similar approach can be applied to the CV model.

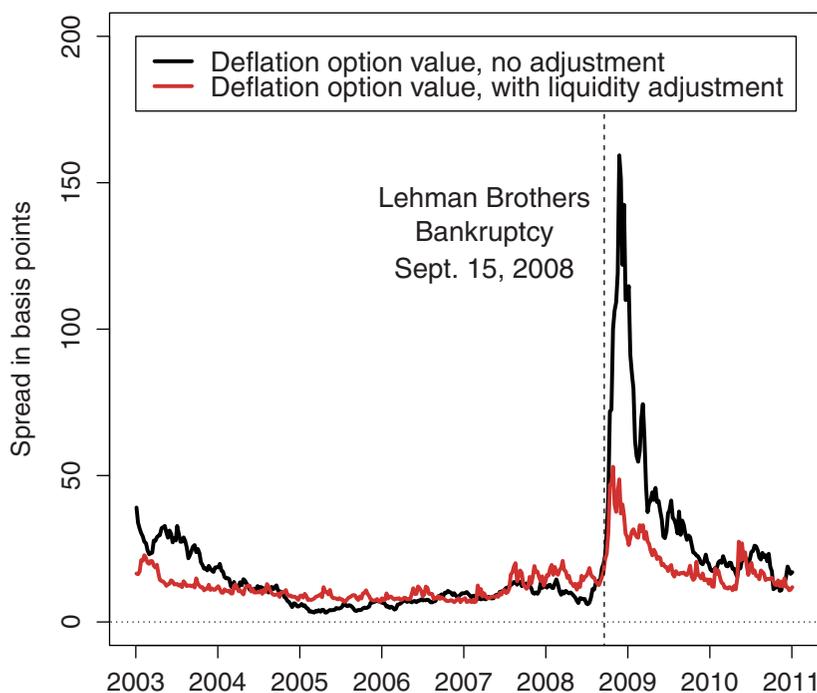
**Table VIII.** Deflation option value regression results, 2003–2010

The top panel reports the results of regressions with the par-bond yield spread between a seasoned and a newly issued 5-year TIPS implied by the CV model, while the bottom panel reports the regression results for the corresponding measure implied by the SV model. *T*-statistics are reported in parentheses. Asterisks \* and \*\* indicate significance at the 5% and 1% levels, respectively. The data samples are weekly covering the period from January 3, 2003, to December 31, 2010, a total of 418 observations. The model-implied deflation option values are based on the estimated parameter values over the full data sample.

Explanatory variables	(1)	(2)	(3)	(4)	(5)	(6)	(7)
CV model							
Constant	-0.12 (-15.41)	-0.05** (-11.60)	-0.08** (-12.37)	-0.03** (-4.41)	0.13** (6.17)	-0.07** (-10.18)	-0.04** (-2.72)
VIX	0.89** (26.46)					0.20** (3.86)	0.26** (4.54)
HPW measure		0.03** (35.75)				0.03** (15.41)	0.04** (12.71)
Off-the-run spread			0.87** (26.93)				-0.19** (-2.62)
AAA credit spread				0.21** (18.41)			-0.06** (-3.67)
TIPS trading volume					-0.01** (-3.12)		-0.00 (-1.37)
Adjusted $R^2$	0.63	0.75	0.63	0.45	0.02	0.76	0.77
Explanatory variables	(1)	(2)	(3)	(4)	(5)	(6)	(7)
SV model							
Constant	-0.17** (-13.37)	-0.01 (-1.91)	-0.09** (-9.01)	0.03* (2.42)	0.39** (10.95)	-0.10** (-8.24)	0.00 (-0.04)
VIX	1.74** (32.41)					0.77** (8.75)	0.67** (7.41)
HPW measure		0.06** (36.11)				0.04** (12.72)	0.04** (10.17)
Off-the-run spread			1.72** (34.60)				0.10 (0.82)
AAA credit spread				0.36** (17.01)			-0.10** (-3.88)
TIPS trading volume					-0.03** (-5.79)		-0.01** (-4.03)
Adjusted $R^2$	0.72	0.76	0.74	0.41	0.07	0.79	0.83

sample and again around the peak of the financial crisis as discussed in CLR. Outside those two relatively short periods, illiquidity effects appear to have mattered less for our analysis.

To generate liquidity-adjusted estimates of the risk of deflation, we assume that nominal yields—as well as the nominal level factor and the common slope and curvature factors ( $L_t^N, S_t, C_t$ )—are approximately free of liquidity effects in comparison to TIPS yields. This assumption implies that (to a first order approximation) the real yield level factor  $L_t^R$  is



**Figure 8.** The liquidity-adjusted 5-year deflation option values.

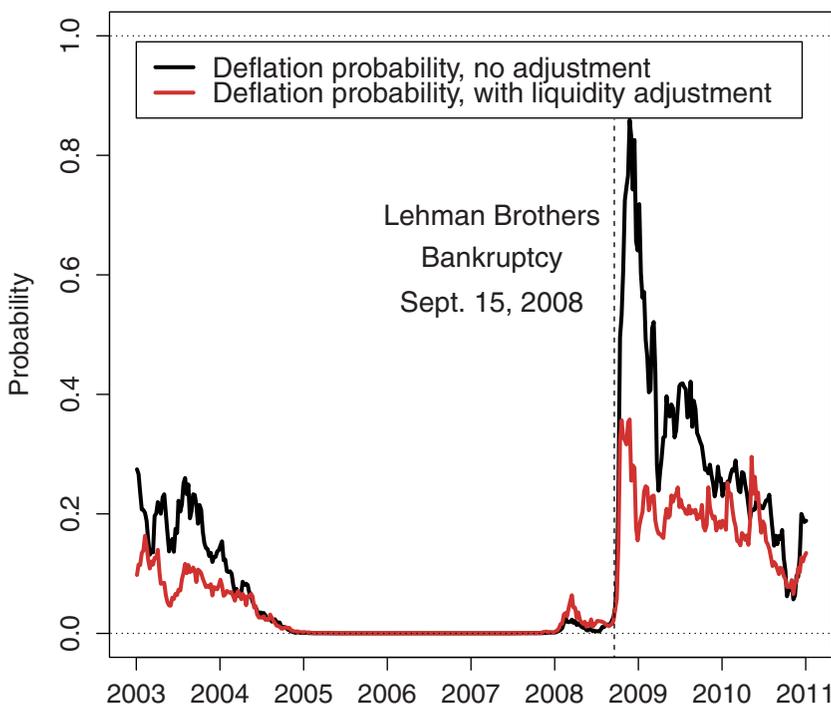
Illustration of the estimated 5-year deflation option values from the SV model with and without adjustment for TIPS liquidity effects. The data series represent full sample weekly estimates covering the period from January 3, 2003, to December 31, 2010.

affected by TIPS illiquidity. As a consequence, we generate an alternative real yield level factor by backing out the alternative path of  $L_t^R$  that makes the model-implied 5-year deflation option values match the liquidity-adjusted option values for each observation over the period from January 3, 2003, to December 31, 2010, while keeping the other three state variables at their estimated values.  $\tilde{L}_t^R$  is used to denote this alternative path for the real yield level factor. Based on the estimated paths of the first three factors and  $\tilde{L}_t^R$ , Figure 9 shows the estimated 1-year deflation probabilities over the full sample period as well as the corresponding estimates without any liquidity corrections.

We note that the largest liquidity adjustments are observed in 2003 and again around the peak of the financial crisis in late 2008 and early 2009. More importantly, though, is the observation that the TIPS illiquidity effects do not alter the inference regarding when the risk of deflation is elevated; i.e., namely the 2003–04 deflation scare and the 2008–09 crisis period. However, the illiquidity effects do induce an upward bias in the assessment of the severity of the deflation risk in these periods of relative market illiquidity. Further research into this aspect of TIPS pricing and liquidity premiums is needed.

## 8. Conclusion

In this paper, we examine the deflation protection option embedded in TIPS over the period from 2003 to 2010, including the depths of the financial crisis in late 2008 and



**Figure 9.** The liquidity-adjusted 1-year deflation probabilities.

Illustration of the estimated 1-year deflation probabilities under the objective probability measure from the SV model with and without adjustment for TIPS liquidity effects. The data series represent full sample weekly estimates covering the period from January 3, 2003, to December 31, 2010.

early 2009. To do so, we modify the joint model of nominal and real bond yields introduced in CLR by replacing its CV assumption with SV driven by the model's nominal and real level factors. Our preferred specification of the SV model delivers reasonable decompositions of BEI (i.e., the spread between nominal and real Treasury yields) into expected inflation and inflation risk premiums, showing that this model captures the data's first moment dynamics as well as the CV model. However, the SV model is shown to be better able to price the value of deflation protection embedded in TIPS and proxied for here by the difference between similar TIPS with differing degrees of accumulated inflation protection. This result highlights that the SV model is better able to capture the volatility dynamics observed in the data and critical to derivatives pricing. Based on this evidence, the proposed SV model should be useful for judging bond investors' views on the tail risk of deflation as well as their inflation expectations. The SV model is an obvious candidate for pricing derivative products in the inflation swap market, a topic we leave for future research.

In analyzing our model-implied deflation option values, our regression results suggest that general economic uncertainty—as measured by the VIX index—is a key driver, but that measures of market illiquidity are also important. We attempted a simple adjustment for such effects combining our model structure with regression results for the deflation option values. However, a more comprehensive correction of TIPS yields for liquidity effects

would be desirable (see Pflueger and Viceira, 2013; D'Amico, Kim, and Wei, 2014; Abrahams *et al.*, 2015 for examples), but this is another topic that we leave for future research.

Finally, we end our sample in 2010 to avoid addressing the problem of the zero lower bound of nominal yields. However, going forward, this will be a critical issue to address as short-term US Treasury yields have been near the zero lower bound since mid-2011. Christensen and Rudebusch (2015) introduce a tractable shadow-rate AFNS model class, which respects the zero lower bound for nominal bond yields and could be explored further. Again, we leave this important endeavor for future research.

## Appendix A: Bond Price Formulas

In the SV model, nominal zero-coupon bond prices are given by

$$P^N(t, T) = E_t^Q[\exp(-\int_t^T r_u^N du)] = \exp(B_1^N(t, T)L_t^N + B_2^N(t, T)S_t + B_3^N(t, T)C_t + B_4^N(t, T)L_t^R + A^N(t, T)),$$

where  $B_1^N(t, T)$ ,  $B_2^N(t, T)$ ,  $B_3^N(t, T)$ , and  $B_4^N(t, T)$  are the unique solutions to the following system of ODEs

$$\begin{pmatrix} \frac{dB_1^N(t, T)}{dt} \\ \frac{dB_2^N(t, T)}{dt} \\ \frac{dB_3^N(t, T)}{dt} \\ \frac{dB_4^N(t, T)}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \kappa_{L^N}^Q & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & \kappa_{L^R}^Q \end{pmatrix} \begin{pmatrix} B_1^N(t, T) \\ B_2^N(t, T) \\ B_3^N(t, T) \\ B_4^N(t, T) \end{pmatrix} \\ - \frac{1}{2} \sum_{j=1}^4 \left[ \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix} \begin{pmatrix} (B_1^N)^2 & B_1^N B_2^N & B_1^N B_3^N & B_1^N B_4^N \\ B_1^N B_2^N & (B_2^N)^2 & B_2^N B_3^N & B_2^N B_4^N \\ B_1^N B_3^N & B_2^N B_3^N & (B_3^N)^2 & B_3^N B_4^N \\ B_1^N B_4^N & B_2^N B_4^N & B_3^N B_4^N & (B_4^N)^2 \end{pmatrix} \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix} \right]_{jj} (\delta^j)',$$

and  $\gamma$  and  $\delta$  are given by

$$\gamma = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This structure implies that the factor loadings in the nominal zero-coupon bond price function are given by the unique solution to the following set of ODEs

$$\frac{dB_1^N(t, T)}{dt} = 1 + \kappa_{LN}^Q B_1^N(t, T) - \frac{1}{2} \sigma_{11}^2 B_1^N(t, T)^2, \quad B_1^N(T, T) = \bar{B}_1^N,$$

$$\frac{dB_2^N(t, T)}{dt} = 1 + \lambda B_2^N(t, T), \quad B_2^N(T, T) = \bar{B}_2^N,$$

$$\frac{dB_3^N(t, T)}{dt} = -\lambda B_2^N(t, T) + \lambda B_3^N(t, T), \quad B_3^N(T, T) = \bar{B}_3^N,$$

$$\frac{dB_4^N(t, T)}{dt} = \kappa_{LR}^Q B_4^N(t, T) - \frac{1}{2} \sigma_{44}^2 B_4^N(t, T)^2, \quad B_4^N(T, T) = \bar{B}_4^N.$$

These four ODEs have the following unique solution<sup>24</sup>:

$$B_1^N(t, T) = \frac{-2 \left[ e^{\phi^N(T-t)} - 1 \right] + \bar{B}_1^N e^{\phi^N(T-t)} (\phi^N - \kappa_{LN}^Q) + \bar{B}_1^N (\phi^N + \kappa_{LN}^Q)}{2\phi^N + (\phi^N + \kappa_{LN}^Q - \bar{B}_1^N \sigma_{11}^2) \left[ e^{\phi^N(T-t)} - 1 \right]},$$

$$B_2^N(t, T) = e^{-\lambda(T-t)} \bar{B}_2^N - \frac{1 - e^{-\lambda(T-t)}}{\lambda},$$

$$B_3^N(t, T) = \lambda(T-t) e^{-\lambda(T-t)} \bar{B}_2^N + \bar{B}_3^N e^{-\lambda(T-t)} + \left[ (T-t) e^{-\lambda(T-t)} - \frac{1 - e^{-\lambda(T-t)}}{\lambda} \right],$$

$$B_4^N(t, T) = \frac{2\kappa_{LR}^Q \bar{B}_4^N}{(2\kappa_{LR}^Q - \bar{B}_4^N \sigma_{44}^2) e^{\kappa_{LR}^Q(T-t)} + \bar{B}_4^N \sigma_{44}^2},$$

Where

$$\phi^N = \sqrt{(\kappa_{LN}^Q)^2 + 2\sigma_{11}^2}.$$

Now, the  $A^N(t, T)$ -function in the yield-adjustment term in the nominal zero-coupon bond yield function is given by the solution to the following ODE:

$$\frac{dA^N(t, T)}{dt} = -B^N(t, T)' K^Q \theta^Q - \frac{1}{2} \sigma_2^2 B_2^N(t, T)^2 - \frac{1}{2} \sigma_3^2 B_3^N(t, T)^2, \quad A^N(T, T) = \bar{A}^N.$$

This ODE has the following unique solution:

$$A^N(t, T) = \bar{A}^N + \frac{2\kappa_{LN}^Q \theta_{LN}^Q}{\sigma_{11}^2} \ln \left[ \frac{2\phi^N e^{\frac{1}{2}(\phi^N + \kappa_{LN}^Q)(T-t)}}{2\phi^N + (\phi^N + \kappa_{LN}^Q - \bar{B}_1^N \sigma_{11}^2)(e^{\phi^N(T-t)} - 1)} \right] \\ + \sigma_{22}^2 \left[ \frac{1}{2\lambda^2} (T-t) - \frac{(1 + \lambda \bar{B}_2^N)}{\lambda^3} \left[ 1 - e^{-\lambda(T-t)} \right] + \frac{(1 + \lambda \bar{B}_2^N)^2}{4\lambda^3} \left[ 1 - e^{-2\lambda(T-t)} \right] \right]$$

24 The calculations leading to this result are available upon request.

$$\begin{aligned}
 & + \sigma_{33}^2 \left[ \frac{1}{2\lambda^2} (T-t) + \frac{1 + \lambda \bar{B}_2^N}{\lambda^2} (T-t) e^{-\lambda(T-t)} - \frac{(1 + \lambda \bar{B}_2^N)^2}{4\lambda} (T-t)^2 e^{-2\lambda(T-t)} \right. \\
 & \quad - \frac{(1 + \lambda \bar{B}_2^N)(3 + \lambda \bar{B}_2^N + 2\lambda \bar{B}_3^N)}{4\lambda^2} (T-t) e^{-2\lambda(T-t)} \\
 & \quad \left. + \frac{(2 + \lambda \bar{B}_2^N + \lambda \bar{B}_3^N)^2 + (1 + \lambda \bar{B}_3^N)^2}{8\lambda^3} [1 - e^{-2\lambda(T-t)}] - \frac{2 + \lambda \bar{B}_2^N + \lambda \bar{B}_3^N}{\lambda^3} [1 - e^{-\lambda(T-t)}] \right] \\
 & + \frac{2\kappa_{LR}^O \theta_{LR}^O}{\sigma_{44}^2} \ln \left[ \frac{2\kappa_{LR}^O e^{\kappa_{LR}^O (T-t)}}{(2\kappa_{LR}^O - \bar{B}_4^N \sigma_{44}^2) e^{\kappa_{LR}^O (T-t)} + \bar{B}_4^N \sigma_{44}^2} \right].
 \end{aligned}$$

In the SV model, the real zero-coupon bond prices are given by

$$P^R(t, T) = E_t^Q \left[ \exp \left( - \int_t^T r_u^R du \right) \right] = \exp(B_1^R(t, T)L_t^N + B_2^R(t, T)S_t + B_3^R(t, T)C_t + B_4^R(t, T)L_t^R + A^R(t, T)),$$

where  $B_1^R(t, T)$ ,  $B_2^R(t, T)$ ,  $B_3^R(t, T)$ , and  $B_4^R(t, T)$  are the unique solutions to the following system of ODEs:

$$\begin{aligned}
 & \begin{pmatrix} \frac{dB_1^R(t, T)}{dt} \\ \frac{dB_2^R(t, T)}{dt} \\ \frac{dB_3^R(t, T)}{dt} \\ \frac{dB_4^R(t, T)}{dt} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^R \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \kappa_{LN}^O & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & \kappa_{LR}^O \end{pmatrix} \begin{pmatrix} B_1^R(t, T) \\ B_2^R(t, T) \\ B_3^R(t, T) \\ B_4^R(t, T) \end{pmatrix} \\
 & - \frac{1}{2} \sum_{j=1}^4 \left[ \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix} \begin{pmatrix} (B_1^R)^2 & B_1^R B_2^R & B_1^R B_3^R & B_1^R B_4^R \\ B_1^R B_2^R & (B_2^R)^2 & B_2^R B_3^R & B_2^R B_4^R \\ B_1^R B_3^R & B_2^R B_3^R & (B_3^R)^2 & B_3^R B_4^R \\ B_1^R B_4^R & B_2^R B_4^R & B_3^R B_4^R & (B_4^R)^2 \end{pmatrix} \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 \\ 0 & 0 & 0 & \sigma_{44} \end{pmatrix} \right]_{jj} (\delta^j)'.
 \end{aligned}$$

This implies that the factor loadings in the real zero-coupon bond price function are given by the unique solution to the following set of ODEs:

$$\begin{aligned}
 \frac{dB_1^R(t, T)}{dt} &= \kappa_{LN}^O B_1^R(t, T) - \frac{1}{2} \sigma_{11}^2 B_1^R(t, T)^2, \quad B_1^R(T, T) = \bar{B}_1^R, \\
 \frac{dB_2^R(t, T)}{dt} &= \alpha^R + \lambda B_2^R(t, T), \quad B_2^R(T, T) = \bar{B}_2^R, \\
 \frac{dB_3^R(t, T)}{dt} &= -\lambda B_2^R(t, T) + \lambda B_3^R(t, T), \quad B_3^R(T, T) = \bar{B}_3^R, \\
 \frac{dB_4^R(t, T)}{dt} &= 1 + \kappa_{LR}^O B_4^R(t, T) - \frac{1}{2} \sigma_{44}^2 B_4^R(t, T)^2, \quad B_4^R(T, T) = \bar{B}_4^R.
 \end{aligned}$$

These four ODEs have the following unique solution<sup>25</sup>:

$$\begin{aligned}
 B_1^R(t, T) &= \frac{2\kappa_{LN}^O \bar{B}_1^R}{(2\kappa_{LN}^O - \bar{B}_1^R \sigma_{11}^2) e^{\kappa_{LN}^O (T-t)} + \bar{B}_1^R \sigma_{11}^2}, \\
 B_2^R(t, T) &= e^{-\lambda(T-t)} \bar{B}_2^R - \alpha^R \frac{1 - e^{-\lambda(T-t)}}{\lambda}, \\
 B_3^R(t, T) &= \lambda(T-t) e^{-\lambda(T-t)} \bar{B}_2^R + \bar{B}_3^R e^{-\lambda(T-t)} + \alpha^R \left[ (T-t) e^{-\lambda(T-t)} - \frac{1 - e^{-\lambda(T-t)}}{\lambda} \right], \\
 B_4^R(t, T) &= \frac{-2 \left[ e^{\phi^R (T-t)} - 1 \right] + \bar{B}_4^R e^{\phi^R (T-t)} (\phi^R - \kappa_{LR}^O) + \bar{B}_4^R (\phi^R + \kappa_{LR}^O)}{2\phi^R + (\phi^R + \kappa_{LR}^O - \bar{B}_4^R \sigma_{44}^2) \left[ e^{\phi^R (T-t)} - 1 \right]},
 \end{aligned}$$

Where

$$\phi^R = \sqrt{(\kappa_{LR}^O)^2 + 2\sigma_{44}^2}.$$

The  $A^R(t, T)$ -function in the yield-adjustment term in the real zero-coupon bond yield function is given by the solution to the following ODE:

$$\frac{dA^R(t, T)}{dt} = -B^R(t, T)' K^O \theta^O - \frac{1}{2} \sigma_2^2 B_2^R(t, T)^2 - \frac{1}{2} \sigma_3^2 B_3^R(t, T)^2, \quad A^R(T, T) = \bar{A}^R,$$

which is

$$\begin{aligned}
 A^R(t, T) &= \bar{A}^R + \frac{2\kappa_{LN}^O \theta_{LN}^O}{\sigma_{11}^2} \ln \left[ \frac{2\kappa_{LN}^O e^{\kappa_{LN}^O (T-t)}}{(2\kappa_{LN}^O - \bar{B}_1^R \sigma_{11}^2) e^{\kappa_{LN}^O (T-t)} + \bar{B}_1^R \sigma_{11}^2} \right] \\
 &+ \sigma_{22}^2 \left[ \frac{(\alpha^R)^2}{2\lambda^2} (T-t) - \alpha^R \frac{(\alpha^R + \lambda \bar{B}_2^R)}{\lambda^3} \left[ 1 - e^{-\lambda(T-t)} \right] + \frac{(\alpha^R + \lambda \bar{B}_2^R)^2}{4\lambda^3} \left[ 1 - e^{-2\lambda(T-t)} \right] \right] \\
 &+ \sigma_{33}^2 \left[ \frac{(\alpha^R)^2}{2\lambda^2} (T-t) + \alpha^R \frac{\alpha^R + \lambda \bar{B}_2^R}{\lambda^2} (T-t) e^{-\lambda(T-t)} - \frac{(\alpha^R + \lambda \bar{B}_2^R)^2}{4\lambda} (T-t)^2 e^{-2\lambda(T-t)} \right. \\
 &\quad \left. - \frac{(\alpha^R + \lambda \bar{B}_2^R)(3\alpha^R + \lambda \bar{B}_2^R + 2\lambda \bar{B}_3^R)}{4\lambda^2} (T-t) e^{-2\lambda(T-t)} \right. \\
 &\quad \left. + \frac{(2\alpha^R + \lambda \bar{B}_2^R + \lambda \bar{B}_3^R)^2 + (\alpha^R + \lambda \bar{B}_3^R)^2}{8\lambda^3} \left[ 1 - e^{-2\lambda(T-t)} \right] \right. \\
 &\quad \left. - \alpha^R \frac{2\alpha^R + \lambda \bar{B}_2^R + \lambda \bar{B}_3^R}{\lambda^3} \left[ 1 - e^{-\lambda(T-t)} \right] \right] \\
 &+ \frac{2\kappa_{LR}^O \theta_{LR}^O}{\sigma_{44}^2} \ln \left[ \frac{2\phi^R e^{\frac{1}{2}(\phi^R + \kappa_{LR}^O)(T-t)}}{2\phi^R + (\phi^R + \kappa_{LR}^O - \bar{B}_4^R \sigma_{44}^2) (e^{\phi^R (T-t)} - 1)} \right].
 \end{aligned}$$

25 The calculations leading to this result are available upon request.

## Appendix B: Calculation of the NPV of the TIPS Principal Deflation Protection

In general, we are interested in finding the NPV of terminal payoffs from TIPS contingent on the cumulated inflation being below some critical value  $q$ , specifically the following difference is of interest:

$$E_t^Q \left[ e^{-\int_t^T r_s^N ds} \mathbf{1}_{\left\{ \frac{\Pi_T}{\Pi_t} \leq 1+q \right\}} \right] - E_t^Q \left[ e^{-\int_t^T r_s^R ds} \mathbf{1}_{\left\{ \frac{\Pi_T}{\Pi_t} \leq 1+q \right\}} \right].$$

Thus, the states of the world of interest are characterized by

$$\frac{\Pi_T}{\Pi_t} \leq 1 + q \Leftrightarrow Y_{t,T} = \int_t^T (r_s^N - r_s^R) ds \leq \ln(1 + q).$$

Since we are pricing, we need the dynamics of the state variables under the  $Q$ -measure

$$\begin{aligned} \begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \\ dY_{0,t} \end{pmatrix} &= \begin{pmatrix} \kappa_{LN}^Q & 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \kappa_{LR}^Q & 0 \\ -1 & -(1 - \alpha^R) & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \begin{pmatrix} \theta_{LN}^Q \\ 0 \\ 0 \\ \theta_{LR}^Q \\ 0 \end{pmatrix} - \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \\ Y_{0,t} \end{pmatrix} \\ dt \end{bmatrix} \\ &+ \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{L_t^N} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{L_t^R} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1} \end{pmatrix} \begin{pmatrix} dW_t^{LN,Q} \\ dW_t^{S,Q} \\ dW_t^{C,Q} \\ dW_t^{LR,Q} \\ dW_t^{Y,Q} \end{pmatrix}, \end{aligned}$$

where  $Z_{0,t} = (L_t^N, S_t, C_t, L_t^R, Y_{0,t})$  represents the augmented state vector.

Now, define the following two intermediate functions:

$$\psi^1(\bar{B}, t, T) = E_t^Q \left[ e^{-\int_t^T r_s^R ds} e^{\bar{B}' Z_{t,T}} \right] \quad \text{and} \quad \psi^2(\bar{B}, t, T) = E_t^Q \left[ e^{-\int_t^T r_s^N ds} e^{\bar{B}' Z_{t,T}} \right].$$

In order to calculate  $\psi^1(\bar{B}, t, T)$  and  $\psi^2(\bar{B}, t, T)$ , we summarize the  $Q$ -dynamics by the following matrices and vectors:

$$K^Q = \begin{pmatrix} \kappa_{LN}^Q & 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \kappa_{LR}^Q & 0 \\ -1 & -(1 - \alpha^R) & 0 & 1 & 0 \end{pmatrix}, \quad \theta^Q = \begin{pmatrix} \theta_{LN}^Q \\ 0 \\ 0 \\ \theta_{LR}^Q \\ 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho^N = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \rho^R = \begin{pmatrix} 0 \\ \alpha^R \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Furthermore,  $\gamma$  and  $\delta$  are given by

$$\gamma = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From Duffie, Pan, and Singleton (2000) it follows that

$$\psi^1(\bar{B}, t, T) = \exp(B_{\psi^1}(t, T)'Z_{t,t} + A_{\psi^1}(t, T)),$$

where  $B_{\psi^1}(t, T)$  and  $A_{\psi^1}(t, T)$  are the solutions to the following system of ODEs:

$$\frac{dB_{\psi^1}(t, T)}{dt} = \rho^R + (K^Q)'B_{\psi^1}(t, T) - \frac{1}{2} \sum_{j=1}^5 (\Sigma' B_{\psi^1}(t, T) B_{\psi^1}(t, T)' \Sigma)_{j,j} (\delta^j)', \quad B_{\psi^1}(T, T) = \bar{B}, \quad (\text{B.1})$$

$$\frac{dA_{\psi^1}(t, T)}{dt} = -B_{\psi^1}(t, T)' K^Q \theta^Q - \frac{1}{2} \sum_{j=1}^5 (\Sigma' B_{\psi^1}(t, T) B_{\psi^1}(t, T)' \Sigma)_{j,j} \gamma^j, \quad A_{\psi^1}(T, T) = 0. \quad (\text{B.2})$$

This system of ODEs can be solved analytically and the solution is provided in the following proposition.

**Proposition 1** *Let the state variables be given by  $Z_{t,T} = (L_t^N, S_t, C_t, L_t^R, Y_{t,T})$ , and let the real instantaneous risk-free rate be given by*

$$r_t^R = (\rho^R)' X_t,$$

then

$$\psi^1(\bar{B}, t, T) = \exp(B_{\psi^1}^1(t, T)L_t^N + B_{\psi^1}^2(t, T)S_t + B_{\psi^1}^3(t, T)C_t + B_{\psi^1}^4(t, T)L_t^R + B_{\psi^1}^5(t, T)Y_{t,t} + A_{\psi^1}(t, T))$$

where<sup>26</sup>

$$B_{\psi^1}^1(t, T) = \frac{-2\rho_1 \left[ e^{\phi_{\psi^1}^N(T-t)} - 1 \right] + \bar{B}^1 \left( \phi_{\psi^1}^N - \kappa_{LN}^Q \right) e^{\phi_{\psi^1}^N(T-t)} + \bar{B}^1 \left( \phi_{\psi^1}^N + \kappa_{LN}^Q \right)}{2\phi_{\psi^1}^N + \left( \phi_{\psi^1}^N + \kappa_{LN}^Q - \bar{B}^1 \sigma_{11}^2 \right) \left[ e^{\phi_{\psi^1}^N(T-t)} - 1 \right]},$$

$$B_{\psi^1}^2(t, T) = e^{-\lambda(T-t)} \bar{B}^2 - \left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 \right] \frac{1 - e^{-\lambda(T-t)}}{\lambda},$$

$$B_{\psi^1}^3(t, T) = e^{-\lambda(T-t)} \bar{B}^3 + \lambda(T-t) e^{-\lambda(T-t)} \bar{B}^2 + \left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 \right] \left\{ (T-t) e^{-\lambda(T-t)} - \frac{1 - e^{-\lambda(T-t)}}{\lambda} \right\}$$

26 The calculations leading to this result are available upon request.

$$B_{\psi^1}^4(t, T) = \frac{-2\rho_4 \left[ e^{\phi_{\psi^1}^R(T-t)} - 1 \right] + \bar{B}^4 \left( \phi_{\psi^1}^R - \kappa_{LR}^Q \right) e^{\phi_{\psi^1}^R(T-t)} + \bar{B}^4 \left( \phi_{\psi^1}^R + \kappa_{LR}^Q \right)}{2\phi_{\psi^1}^R + \left( \phi_{\psi^1}^R + \kappa_{LR}^Q - \bar{B}^4 \sigma_{44}^2 \right) \left[ e^{\phi_{\psi^1}^R(T-t)} - 1 \right]},$$

$$B_{\psi^1}^5(t, T) = \bar{B}^5,$$

and

$$\begin{aligned} A_{\psi^1}(t, T) = & \frac{2\kappa_{LN}^Q \theta_{LN}^Q}{\sigma_{11}^2} \ln \left[ \frac{2\phi_{\psi^1}^N e^{\frac{1}{2}(\phi_{\psi^1}^N + \kappa_{LN}^Q)(T-t)}}{2\phi_{\psi^1}^N + (\phi_{\psi^1}^N + \kappa_{LN}^Q - \bar{B}^1 \sigma_{11}^2) \left[ e^{\phi_{\psi^1}^N(T-t)} - 1 \right]} \right] \\ & + \sigma_{22}^2 \left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 + \lambda \bar{B}^2 \right]^2 \frac{1 - e^{-2\lambda(T-t)}}{4\lambda^3} + \frac{\sigma_{22}^2 \left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 \right]^2}{2\lambda^2} (T-t) \\ & - \sigma_{22}^2 \left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 + \lambda \bar{B}^2 \right] \left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 \right] \frac{1 - e^{-\lambda(T-t)}}{\lambda^3} \\ & + \sigma_{33}^2 \left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 + \lambda \bar{B}^3 \right]^2 \frac{1 - e^{-2\lambda(T-t)}}{4\lambda^3} + \frac{\sigma_{33}^2 \left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 \right]^2}{2\lambda^2} (T-t) \\ & + \frac{\sigma_{33}^2}{2} \left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 + \lambda \bar{B}^2 \right]^2 \left[ -\frac{1}{2\lambda} (T-t)^2 e^{-2\lambda(T-t)} - \frac{1}{2\lambda^2} (T-t) e^{-2\lambda(T-t)} + \frac{1 - e^{-2\lambda(T-t)}}{4\lambda^3} \right] \\ & - \sigma_{33}^2 \left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 + \lambda \bar{B}^3 \right] \left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 \right] \frac{1 - e^{-\lambda(T-t)}}{\lambda^3} \\ & + \sigma_{33}^2 \frac{\left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 + \lambda \bar{B}^3 \right] \left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 + \lambda \bar{B}^2 \right]}{\lambda} \left[ -\frac{1}{2\lambda} (T-t) e^{-2\lambda(T-t)} + \frac{1 - e^{-2\lambda(T-t)}}{4\lambda^2} \right] \\ & - \sigma_{33}^2 \frac{\left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 \right] \left[ \alpha^R - (1 - \alpha^R) \bar{B}^5 + \lambda \bar{B}^2 \right]}{\lambda} \left[ -\frac{1}{\lambda} (T-t) e^{-\lambda(T-t)} + \frac{1 - e^{-\lambda(T-t)}}{\lambda^2} \right] \\ & + \frac{2\kappa_{LR}^Q \theta_{LR}^Q}{\sigma_{44}^2} \ln \left[ \frac{2\phi_{\psi^1}^R e^{\frac{1}{2}(\phi_{\psi^1}^R + \kappa_{LR}^Q)(T-t)}}{2\phi_{\psi^1}^R + (\phi_{\psi^1}^R + \kappa_{LR}^Q - \bar{B}^4 \sigma_{44}^2) \left[ e^{\phi_{\psi^1}^R(T-t)} - 1 \right]} \right] \end{aligned}$$

with

$$\phi_{\psi^1}^N = \sqrt{(\kappa_{LN}^Q)^2 + 2\rho_1 \sigma_{11}^2}, \quad \rho_1 = -\bar{B}^5, \quad \phi_{\psi^1}^R = \sqrt{(\kappa_{LR}^Q)^2 + 2\rho_4 \sigma_{44}^2}, \quad \text{and} \quad \rho_4 = 1 + \bar{B}^5.$$

Using a similar approach, it holds that

$$\psi^2(\bar{B}, t, T) = \exp(B_{\psi^2}(t, T)' Z_{t,T} + A_{\psi^2}(t, T)),$$

where  $B_{\psi^2}(t, T)$  and  $A_{\psi^2}(t, T)$  are the solutions to the following system of ODEs:

$$\frac{dB_{\psi^2}(t, T)}{dt} = \rho^N + (K^Q)' B_{\psi^2}(t, T) - \frac{1}{2} \sum_{j=1}^5 (\Sigma' B_{\psi^2}(t, T) B_{\psi^2}(t, T)' \Sigma)_{ij} (\delta^j)', \quad B_{\psi^2}(T, T) = \bar{B},$$

$$\frac{dA_{\psi^2}(t, T)}{dt} = -B_{\psi^2}(t, T)' K^Q \theta^Q - \frac{1}{2} \sum_{j=1}^5 (\Sigma' B_{\psi^2}(t, T) B_{\psi^2}(t, T)' \Sigma)_{ij} \gamma^j, \quad A_{\psi^2}(T, T) = 0.$$

This system can also be solved analytically and the solution is provided in the following proposition.

**Proposition 2** *Let the state variables be given by  $Z_{t,T} = (L_t^N, S_t, C_t, L_t^R, Y_{t,T})$ , and let the nominal instantaneous risk-free rate be given by*

$$r_t^N = (\rho^N)'X_t,$$

then

$$\psi^2(\bar{B}, t, T) = \exp(B_{\psi^2}^1(t, T)L_t^N + B_{\psi^2}^2(t, T)S_t + B_{\psi^2}^3(t, T)C_t + B_{\psi^2}^4(t, T)L_t^R + B_{\psi^2}^5(t, T)Y_{t,T} + A_{\psi^2}(t, T)),$$

where<sup>27</sup>

$$B_{\psi^2}^1(t, T) = \frac{-2\rho_1[e^{\phi_{\psi^2}^N(T-t)} - 1] + \bar{B}^1(\phi_{\psi^2}^N - \kappa_{L^N}^O)e^{\phi_{\psi^2}^N(T-t)} + \bar{B}^1(\phi_{\psi^2}^N + \kappa_{L^N}^O)}{2\phi_{\psi^2}^N + (\phi_{\psi^2}^N + \kappa_{L^N}^O - \bar{B}^1\sigma_{11}^2)[e^{\phi_{\psi^2}^N(T-t)} - 1]},$$

$$B_{\psi^2}^2(t, T) = e^{-\lambda(T-t)}\bar{B}^2 - [1 - (1 - \alpha^R)\bar{B}^5]\frac{1 - e^{-\lambda(T-t)}}{\lambda},$$

$$B_{\psi^2}^3(t, T) = e^{-\lambda(T-t)}\bar{B}^3 + \lambda(T-t)e^{-\lambda(T-t)}\bar{B}^2 + [1 - (1 - \alpha^R)\bar{B}^5]\left\{(T-t)e^{-\lambda(T-t)} - \frac{1 - e^{-\lambda(T-t)}}{\lambda}\right\}$$

$$B_{\psi^2}^4(t, T) = \frac{-2\rho_4[e^{\phi_{\psi^2}^R(T-t)} - 1] + \bar{B}^4(\phi_{\psi^2}^R - \kappa_{L^R}^O)e^{\phi_{\psi^2}^R(T-t)} + \bar{B}^4(\phi_{\psi^2}^R + \kappa_{L^R}^O)}{2\phi_{\psi^2}^R + (\phi_{\psi^2}^R + \kappa_{L^R}^O - \bar{B}^4\sigma_{44}^2)[e^{\phi_{\psi^2}^R(T-t)} - 1]},$$

$$B_{\psi^2}^5(t, T) = \bar{B}^5,$$

and

$$\begin{aligned} A_{\psi^2}(t, T) &= \frac{2\kappa_{L^N}^O\theta_{L^N}^O}{\sigma_{11}^2} \ln \left[ \frac{2\phi_{\psi^2}^N e^{\frac{1}{2}(\phi_{\psi^2}^N + \kappa_{L^N}^O)(T-t)}}{2\phi_{\psi^2}^N + (\phi_{\psi^2}^N + \kappa_{L^N}^O - \bar{B}^1\sigma_{11}^2)[e^{\phi_{\psi^2}^N(T-t)} - 1]} \right] \\ &+ \sigma_{22}^2 \left[ 1 - (1 - \alpha^R)\bar{B}^5 + \lambda\bar{B}^2 \right]^2 \frac{1 - e^{-2\lambda(T-t)}}{4\lambda^3} + \frac{\sigma_{22}^2}{2} \frac{[1 - (1 - \alpha^R)\bar{B}^5]^2}{\lambda^2} (T-t) \\ &- \sigma_{22}^2 \left[ 1 - (1 - \alpha^R)\bar{B}^5 + \lambda\bar{B}^2 \right] \left[ 1 - (1 - \alpha^R)\bar{B}^5 \right] \frac{1 - e^{-\lambda(T-t)}}{\lambda^3} \\ &+ \sigma_{33}^2 \left[ 1 - (1 - \alpha^R)\bar{B}^5 + \lambda\bar{B}^3 \right]^2 \frac{1 - e^{-2\lambda(T-t)}}{4\lambda^3} + \frac{\sigma_{33}^2}{2} \frac{[1 - (1 - \alpha^R)\bar{B}^5]^2}{\lambda^2} (T-t) \\ &+ \frac{\sigma_{33}^2}{2} \left[ 1 - (1 - \alpha^R)\bar{B}^5 + \lambda\bar{B}^3 \right]^2 \left[ -\frac{1}{2\lambda}(T-t)^2 e^{-2\lambda(T-t)} - \frac{1}{2\lambda^2}(T-t)e^{-2\lambda(T-t)} + \frac{1 - e^{-2\lambda(T-t)}}{4\lambda^3} \right] \\ &- \sigma_{33}^2 \left[ 1 - (1 - \alpha^R)\bar{B}^5 + \lambda\bar{B}^3 \right] \left[ 1 - (1 - \alpha^R)\bar{B}^5 \right] \frac{1 - e^{-\lambda(T-t)}}{\lambda^3} \\ &+ \sigma_{33}^2 \frac{[1 - (1 - \alpha^R)\bar{B}^5 + \lambda\bar{B}^3] \left[ 1 - (1 - \alpha^R)\bar{B}^5 + \lambda\bar{B}^2 \right]}{\lambda} \left[ -\frac{1}{2\lambda}(T-t)e^{-2\lambda(T-t)} + \frac{1 - e^{-2\lambda(T-t)}}{4\lambda^2} \right] \\ &- \sigma_{33}^2 \frac{[1 - (1 - \alpha^R)\bar{B}^5] \left[ 1 - (1 - \alpha^R)\bar{B}^5 + \lambda\bar{B}^2 \right]}{\lambda} \left[ -\frac{1}{\lambda}(T-t)e^{-\lambda(T-t)} + \frac{1 - e^{-\lambda(T-t)}}{\lambda^2} \right] \\ &+ \frac{2\kappa_{L^R}^O\theta_{L^R}^O}{\sigma_{44}^2} \ln \left[ \frac{2\phi_{\psi^2}^R e^{\frac{1}{2}(\phi_{\psi^2}^R + \kappa_{L^R}^O)(T-t)}}{2\phi_{\psi^2}^R + (\phi_{\psi^2}^R + \kappa_{L^R}^O - \bar{B}^4\sigma_{44}^2)[e^{\phi_{\psi^2}^R(T-t)} - 1]} \right] \end{aligned}$$

27 The calculations leading to this result are available upon request.

with

$$\phi_{\psi^2}^N = \sqrt{(\kappa_{LN}^Q)^2 + 2\rho_1\sigma_{11}^2}, \quad \rho_1 = 1 - \bar{B}^5, \quad \phi_{\psi^2}^R = \sqrt{(\kappa_{LR}^Q)^2 + 2\rho_4\sigma_{44}^2}, \quad \text{and} \quad \rho_4 = \bar{B}^5.$$

With these results at our disposal, we can turn our attention to the pricing of the deflation protection option in the TIPS contract. From [Duffie, Pan, and Singleton \(2000\)](#) it follows that

$$E_t^Q \left[ e^{-\int_t^T r_s^R ds} e^{\bar{B}'Z_{t,T}} \mathbf{1}_{\{b'Z_{t,T} \leq z\}} \right] = \frac{\psi^1(\bar{B}, t, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}\{e^{-ivz} \psi^1(\bar{B} + ivb, t, T)\}}{v} dv,$$

$$E_t^Q \left[ e^{-\int_t^T r_s^N ds} e^{\bar{B}'Z_{t,T}} \mathbf{1}_{\{b'Z_{t,T} \leq z\}} \right] = \frac{\psi^2(\bar{B}, t, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}\{e^{-ivz} \psi^2(\bar{B} + ivb, t, T)\}}{v} dv.$$

Since we are interested in the condition

$$Y_{t,T} = \int_t^T (r_s^N - r_s^R) ds \leq \ln(1 + q),$$

the expectations above should be evaluated at  $b = (0, 0, 0, 0, 1)$ ,  $z = \ln(1 + q)$ , and  $\bar{B} = (0, 0, 0, 0, 0)$ .

A similar approach can be used to calculate the NPV of the TIPS deflation protection option within the CV model (see [Christensen, Lopez, and Rudebusch \(2012\)](#) for details).

The functions  $\frac{\text{Im}\{e^{-ivz} \psi^1(\bar{B} + ivb, t, T)\}}{v}$  and  $\frac{\text{Im}\{e^{-ivz} \psi^2(\bar{B} + ivb, t, T)\}}{v}$  that need to be integrated in order to calculate the NPV of the TIPS deflation protection option have already converged to zero for values of  $v$  above 500, so we approximate the infinite integral in the pricing formulas by capping  $v$  at 1,000 to err on the side of conservatism and use a step size of  $\Delta v = 0.01$  in the numerical approximation, which is sufficient since the functions are clearly smooth.

## Appendix C: Deflation Probabilities within the SV Model

[Christensen, Lopez, and Rudebusch \(2012\)](#) use the CV model to generate deflation probabilities at various horizons appropriate for macroeconomic and monetary policy purposes. Similarly, the SV model can be used to calculate deflation probabilities, although additional steps are necessary. The change in the market-implied price index for the period from  $t$  until  $t + \tau$  is given by

$$\frac{\Pi_{t+\tau}}{\Pi_t} = e^{\int_t^{t+\tau} (r_s^N - r_s^R) ds}.$$

We want to calculate the probability of the event that the change in the price index is below a certain critical level  $q$ . By implication, we are interested in the states of the world where

$$\frac{\Pi_{t+\tau}}{\Pi_t} \leq 1 + q,$$

or, equivalently,

$$\int_t^{t+\tau} (r_s^N - r_s^R) ds \leq \ln(1 + q).$$

Since the nominal and real instantaneous short rates are given by

$$r_t^N = L_t^N + S_t,$$

$$r_t^R = L_t^R + \alpha^R S_t,$$

we are interested in the distributional properties of the following process:

$$Y_{0,t} = \int_0^t (r_s^N - r_s^R) ds = \int_0^t (L_s^N + S_s - L_s^R - \alpha^R S_s) ds \Rightarrow dY_{0,t} = (L_t^N + (1 - \alpha^R) S_t - L_t^R) dt.$$

In general, the  $P$ -dynamics of the state variables  $X_t$  are given by

$$dX_t = K^P(\theta^P - X_t)dt + \Sigma D(X_t)dW_t^P.$$

Adding the  $Y_t$ -process to this system, leaves us with a five-factor SDE of the following form:

$$\begin{pmatrix} dL_t^N \\ dS_t \\ dC_t \\ dL_t^R \\ dY_{0,t} \end{pmatrix} = \begin{pmatrix} \kappa_{11}^P & 0 & 0 & \kappa_{14}^P & 0 \\ \kappa_{21}^P & \kappa_{22}^P & \kappa_{23}^P & \kappa_{24}^P & 0 \\ \kappa_{31}^P & \kappa_{32}^P & \kappa_{33}^P & \kappa_{34}^P & 0 \\ \kappa_{41}^P & 0 & 0 & \kappa_{44}^P & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_1^P \\ \theta_2^P \\ \theta_3^P \\ \theta_4^P \\ 0 \end{pmatrix} dt - \begin{pmatrix} \kappa_{11}^P & 0 & 0 & \kappa_{14}^P & 0 \\ \kappa_{21}^P & \kappa_{22}^P & \kappa_{23}^P & \kappa_{24}^P & 0 \\ \kappa_{31}^P & \kappa_{32}^P & \kappa_{33}^P & \kappa_{34}^P & 0 \\ \kappa_{41}^P & 0 & 0 & \kappa_{44}^P & 0 \\ -1 & -(1 - \alpha^R) & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} L_t^N \\ S_t \\ C_t \\ L_t^R \\ Y_{0,t} \end{pmatrix} dt$$

$$+ \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ 0 & 0 & 0 & \sigma_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{L_t^N} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{L_t^R} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} dW_t^{L^N,P} \\ dW_t^{S,P} \\ dW_t^{C,P} \\ dW_t^{L^R,P} \\ dW_t^{Y,P} \end{pmatrix},$$

where  $Z_{0,t} = (L_t^N, S_t, C_t, L_t^R, Y_{0,t})$  represents the augmented state vector.

This is a system of non-Gaussian state variables. As a consequence, we cannot use the approach detailed in Christensen, Lopez, and Rudebusch (2012). Instead, we use the Fourier transform analysis described in full generality for affine models in Duffie, Pan, and Singleton (2000). They provide a formula for calculating contingent expectations of the form

$$G_{\bar{B},\bar{b}}^-(y; Z_{t,t}, t, T) = E^P \left[ e^{-\int_t^T \rho_{\psi}^{\prime} Z_{s,T} ds} e^{\bar{B}^{\prime} Z_{t,T}} \mathbf{1}_{\{\bar{b} Z_{t,T} \leq y\}} | \mathcal{F}_t \right].$$

If we define

$$\psi(\bar{B}; Z_{t,t}, t, T) = E^P \left[ e^{-\int_t^T \rho_{\psi}^{\prime} Z_{s,T} ds} e^{\bar{B}^{\prime} Z_{t,T}} \right] = e^{B_{\psi}(t,T)^{\prime} Z_{t,t} + A_{\psi}(t,T)},$$

where  $B_\psi(t, T)$  and  $A_\psi(t, T)$  are solutions to a system of ODEs similar to the one outlined in Equations (B.1) and (B.2),<sup>28</sup> then Duffie, Pan, and Singleton (2000) show that

$$G_{\bar{B}, \bar{b}}(y; Z_{t,t}, t, T) = \frac{\psi(\bar{B}; Z_{t,t}, t, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}[e^{-iy\gamma} \psi(\bar{B} + iv\bar{b}; Z_{t,t}, t, T)]}{v} dv.$$

Here, we are interested in the cumulative probability function of  $Y_{t,T}$  conditional on  $Z_{t,t}$ , that is, we are interested in the function  $E^P[\mathbf{1}_{\{Y_{t,T} \leq y\}} | \mathcal{F}_t]$ . From the result above it follows that we get the desired probability function if we fix

$$\bar{B} = 0, \quad \bar{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \rho_\psi = 0, \quad \text{and} \quad y = \ln(1 + q).$$

### Priced Deflation Probabilities within the SV Model

The actual probability of deflation calculated above is determined by the estimated factor dynamics under the  $P$ -measure. Thus, it reflects the actual time series dynamics of the state variables. The priced probability of deflation, on the other hand, reflects the implicit probability of deflation needed to match the observed bond prices. Due to risk premia that reflect bond investor risk aversion, this measure can be different from the actual deflation probability. To calculate the priced probability of deflation, we replace the  $P$ -dynamics above with the  $Q$ -dynamics.

### Appendix D: Alternative SV Specifications

In this appendix, we consider alternative ways of introducing SV into the CV model. Specifically, we consider the seven admissible combinations of allowing for spanned SV generated by one or two factors in the model following the work of Christensen, Lopez, and Rudebusch (2014a).

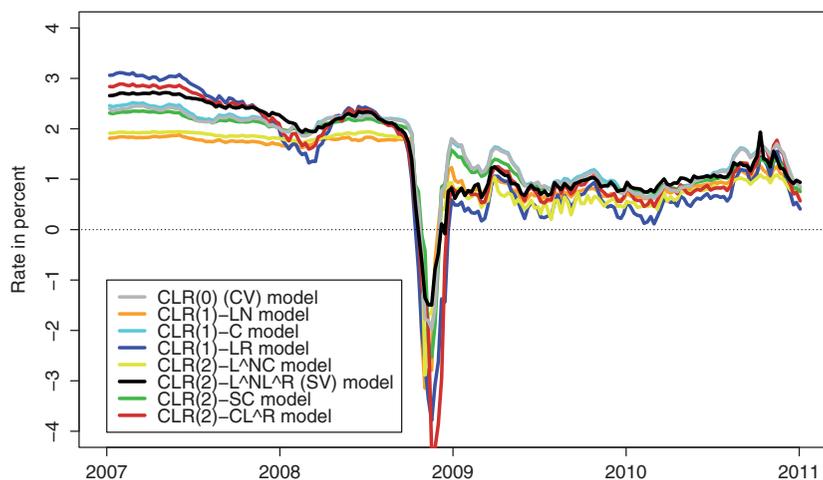
We refer to these models as CLR models because they share the key properties of the CV model as introduced in CLR. First, there are four state variables, which represent a level factor unique to nominal and real yields, respectively, in addition to a slope and curvature factor common to both yield curves. Second, these four state variables have joint dynamics under the risk-neutral probability measure used for pricing closely matching the AFNS model introduced in CDR. Third, the nominal and real short rates are defined as in CLR. To keep the notation simple, we use CLR( $i$ ) to denote a model as defined above with  $i$  referring to the number of factors generating SV, while letters— $L^N$ ,  $S$ ,  $C$ , and  $L^R$ —are used to indicate the source(s) of SV in the model.

28 Note, however, that the solutions differ from the formulas in Appendix B as we are now working under the  $P$ -measure. Thus, we rely on numerical approximations based on a fourth order Runge–Kutta method.

**Table A1.** Summary statistics of the fitted errors

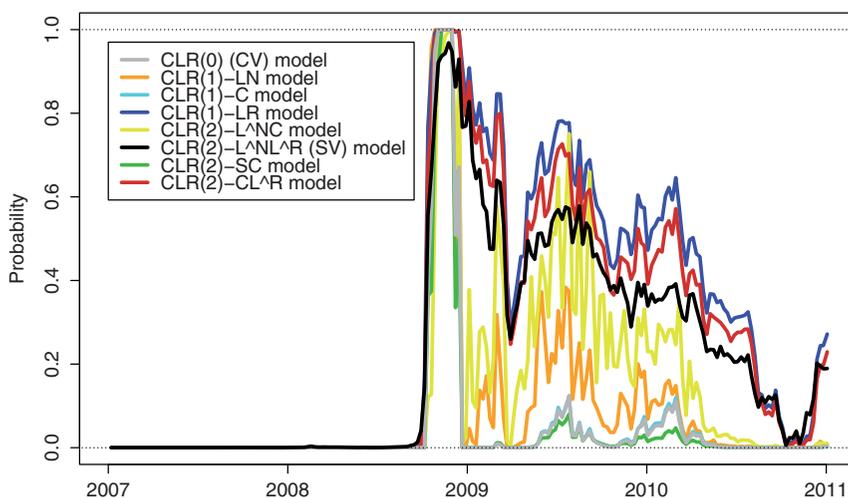
The mean and RMSEs for the preferred specification of each model class are shown. All numbers are measured in basis points. The nominal yields cover the period from January 6, 1995, to December 31, 2010, while the real TIPS yields cover the period from January 3, 2003, to December 31, 2010.

Maturity in months		CLR(0)		CLR(1)-L <sup>N</sup>		CLR(1)-C		CLR(1)-L <sup>R</sup>	
Nominal yields	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	
3	-0.54	9.53	-0.34	19.21	-0.55	9.54	-2.99	20.28	
6	0.00	0.00	-0.77	8.26	0.00	0.00	-2.26	9.07	
12	1.79	5.80	0.00	0.00	1.80	5.81	0.00	0.00	
24	2.22	3.98	0.69	1.63	2.22	3.99	1.44	2.33	
36	0.00	0.13	-0.01	0.08	0.00	0.09	0.00	0.00	
60	-2.67	3.73	-0.59	1.48	-2.67	3.74	-2.10	3.09	
84	0.08	3.37	0.31	0.98	0.10	3.40	0.36	2.99	
120	9.53	12.03	-0.07	4.20	9.57	12.08	8.87	11.46	
TIPS yields	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	
60	-3.98	20.27	-10.70	19.34	-3.49	20.07	-1.14	18.81	
72	-2.60	12.23	-5.16	8.88	-2.27	12.09	-0.02	11.00	
84	-1.31	5.64	0.00	0.00	-1.14	5.57	0.31	4.96	
96	0.00	0.00	4.84	7.73	0.00	0.00	0.00	0.00	
108	1.35	4.94	9.38	14.55	1.20	4.87	-0.83	4.26	
120	2.74	9.32	13.63	20.64	2.46	9.18	-2.08	8.05	
Max logL	52,558.84		53,329.49		52,584.17		52,825.60		
Maturity in months		CLR(2)-L <sup>NC</sup>		CLR(2)-L <sup>NL</sup> <sup>R</sup>		CLR(2)-SC		CLR(2)-CL <sup>R</sup>	
Nominal yields	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	
3	0.44	9.15	0.75	19.23	-0.56	9.54	-0.52	9.53	
6	0.00	0.00	-0.17	8.23	0.00	0.00	0.00	0.00	
12	0.64	5.06	0.00	0.00	1.81	5.82	1.76	5.79	
24	0.98	2.95	0.46	1.56	2.24	4.01	2.18	3.95	
36	0.01	0.34	0.00	0.00	0.00	0.08	0.00	0.00	
60	-0.75	1.80	-0.28	1.27	-2.69	3.78	-2.53	3.57	
84	0.24	1.02	0.24	0.59	0.09	3.43	0.39	3.30	
120	0.10	4.26	-1.15	4.41	9.60	12.05	10.04	12.52	
TIPS yields	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	
60	-12.90	25.74	-2.04	13.59	-3.39	20.03	-0.66	18.69	
72	-8.40	15.76	-0.51	5.87	-2.21	12.06	0.25	10.93	
84	-4.11	7.33	0.00	0.00	-1.12	5.55	0.43	4.93	
96	0.00	0.00	-0.38	4.72	0.00	0.00	0.00	0.00	
108	3.91	6.46	-1.52	8.74	1.18	4.85	-0.92	4.24	
120	7.61	12.22	-3.32	12.35	2.42	9.16	-2.26	8.02	
Max logL	53,520.53		54,470.80		52,563.86		52,851.89		



**Figure A1.** Estimated 5-year inflation expectations.

Illustration of the estimated inflation expectations at the 5-year horizon under the objective  $P$  probability measure according to the eight CLR( $i$ ) models described in the text.

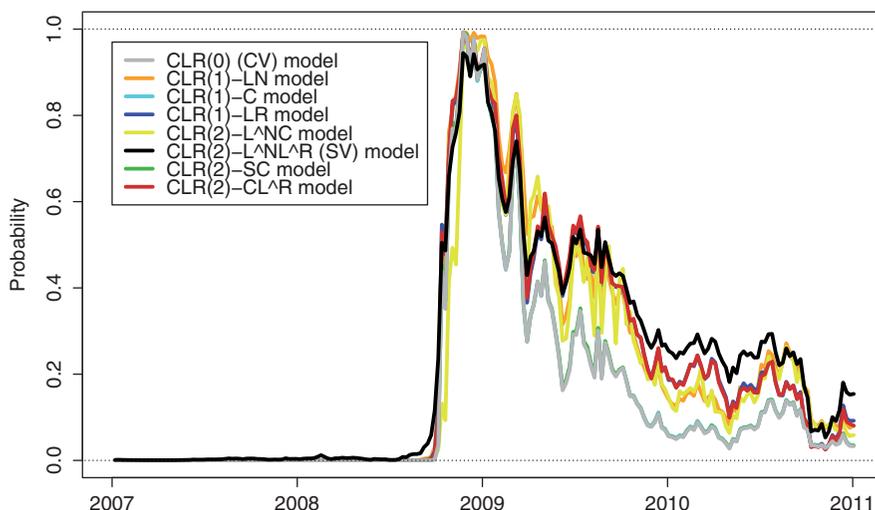


**Figure A2.** Estimated 1-year objective deflation probabilities.

Illustration of the estimated probabilities of deflation over the following year under the objective  $P$  probability measure according to the eight CLR( $i$ ) models described in the text.

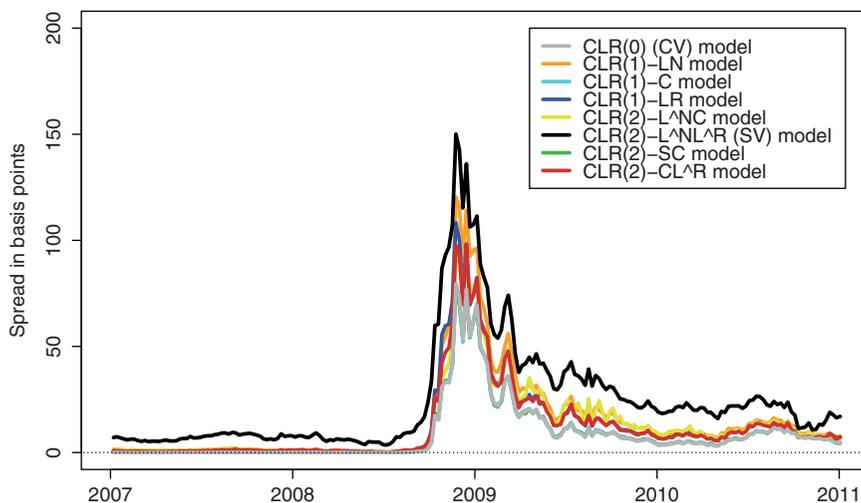
For each model class we go through a careful model selection process similar to the one described in Section 4 to find a preferred specification.<sup>29</sup> We then evaluate the models based on their fit to the data, their model-implied inflation expectations, and their value of the deflation option as defined in the paper. The purpose is to demonstrate that the SV

<sup>29</sup> These results are available from the authors upon request.



**Figure A3.** Estimated 1-year risk-neutral deflation probabilities.

Illustration of the estimated probabilities of deflation over the following year under the risk-neutral  $Q$  probability measure according to the eight CLR(i) models described in the text.



**Figure A4.** Five-year deflation option values.

Illustration of the estimated 5-year deflation option values from the eight CLR(i) models described in the text. The data series represent real-time weekly estimates covering the period from January 6, 2007, to December 31, 2010.

model—the CLR(2)- $L^N L^R$  model in the notation in this Appendix—is competitive relative to both the CV model and alternative CLR(i) models with spanned SV.

The summary statistics of the model fit are reported in Table A1. We note that, when we make the incremental refinement of moving from the Gaussian CLR(0) model (i.e., the CV

model) to models with a single factor generating the SV, the largest improvement in the log likelihood value is obtained when the nominal level factor is allowed to be the source of the SV. On the other hand, letting the curvature factor generate SV delivers only modest improvements. Importantly, this pattern is preserved when we move from one to two SV factors. As a consequence, the model with the nominal and real level factors as the sources of the SV produces the highest log likelihood value among all eight model classes considered.

Figure A1 shows the estimated 5-year expected inflation series from all eight models. We note that they produce rather similar dynamics for the first moment of the inflation distribution. Thus, there is little loss from limiting the focus to Gaussian models provided the main objective is to generate projections of the expected path for inflation consistent with the information reflected in the markets for Treasuries and TIPS. Based on the similarity in the estimates across models we also conclude that the CV and SV models are representative of the results across the eight different model classes in this regard.

Once we focus on tail events such as the risk of net deflation over the coming year, this changes, however, and we see much greater dispersion across models. This is illustrated in Figure A2, which shows the probabilities of net deflation over the next year estimated under the objective probability measure. We note that several of the models are little different from the Gaussian CLR(0) (CV) model. Again, the model with the nominal and real level factors as the sources of the SV stands out as it is clearly one of the models that generate the most variation in the estimated deflation probabilities.<sup>30</sup> Furthermore, it is clear that the conclusion that deflation risk was negligible in the years before the financial crisis is very robust and not sensitive to the specification of SV.

Figure A3 shows the estimated 1-year probabilities of net deflation under the risk-neutral probability measure. Here, we note that the dispersion is smaller under the risk-neutral probability measure than under the objective probability measure. The intuition behind this result is that all the models are estimated to match the observed yields. Thus, for pricing purposes, there is a limit to how different the models can be, which is not the case when it comes to the time-series properties under the objective probability measure.

Finally, Figure A4 shows the estimates of the 5-year deflation option value defined as the par-bond yield spread between a seasoned and a comparable newly issued 5-year TIPS where the deflation protection option value can be assumed to be zero for former and at-the-money for the latter. We note that the range of estimated 5-year deflation option values from the eight models are bounded from below by the estimate from the CV model and from above by the estimate from the SV model. Thus, our choice to focus on the CLR(2)- $L^{NL^R}$  (SV) model creates the greatest contrast in this analysis.

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30 The CLR(1)- $L^R$  model produces the largest estimated objective deflation probabilities in the post-crisis period mainly because its estimated expected inflation is lower and closer to zero than the estimate from the other models. As noted in Figure A3, this does not translate into notably higher estimated deflation probabilities under the risk-neutral probability measure.

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